5.2. Exercises on differentiation

Exercise 1:
Average and instantaneous velocity
Determine graphically
a) the average velocities

$\overline{v}_{[0;1]}$ = $\overline{v}_{[1;3]}$ =

$\overline{v}_{[3;6]}$ = $\overline{v}_{[8;9]}$ =

$\overline{v}_{[10;12]}$ = $\overline{v}_{[14;16]}$ =

$\overline{v}_{[15;17]}$ = $\overline{v}_{[16;18]}$ =

b) the instantaneous velocities

v(0) = v(1) =

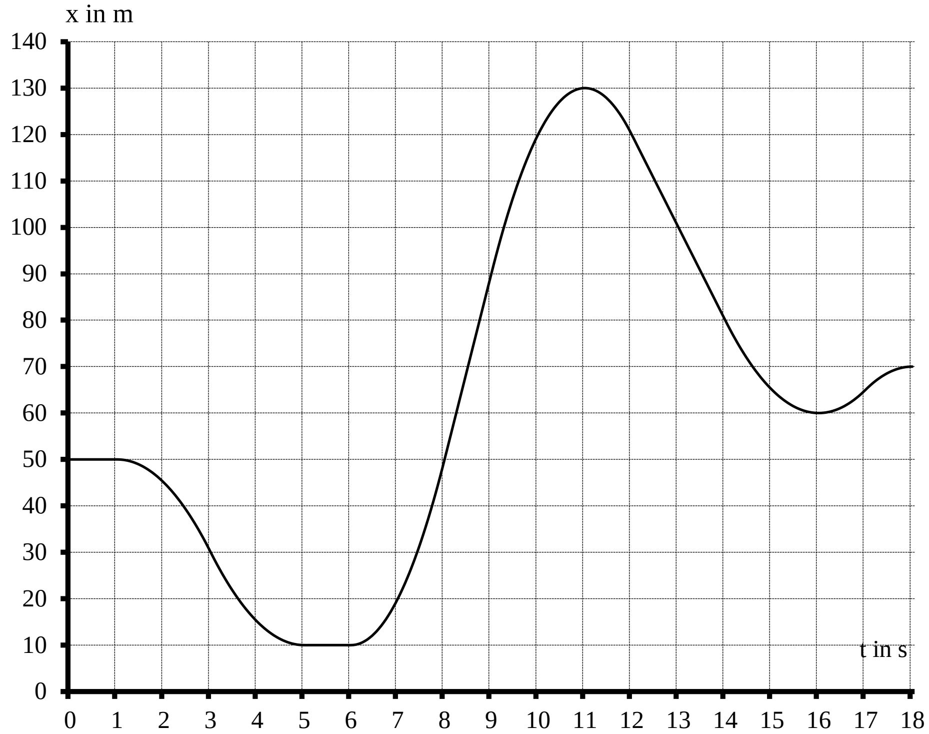
v(2) = v(3) =

v(4) = v(5) =

v(9) = v(19) =

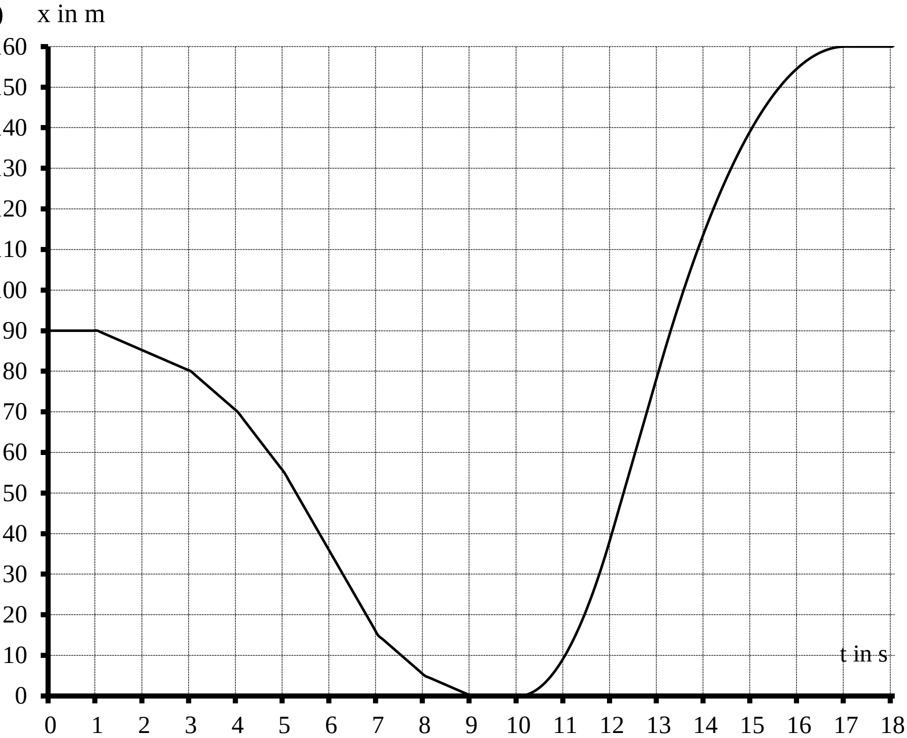
v(11) = v(12) =

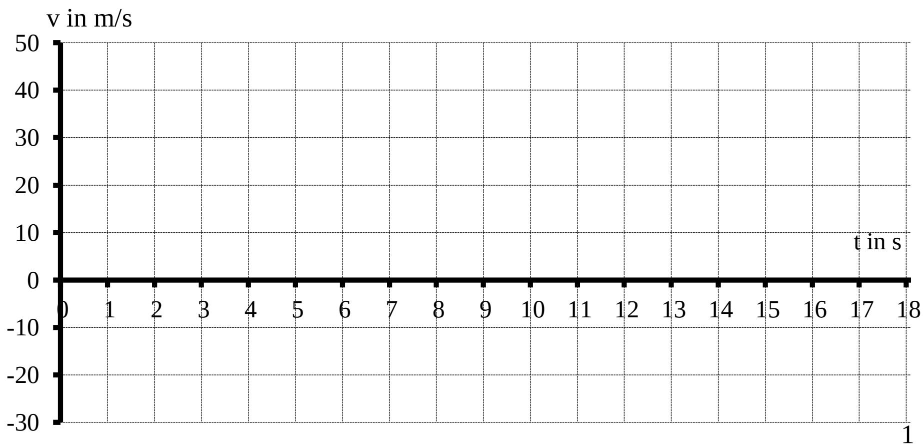
v(14) = v(16) =



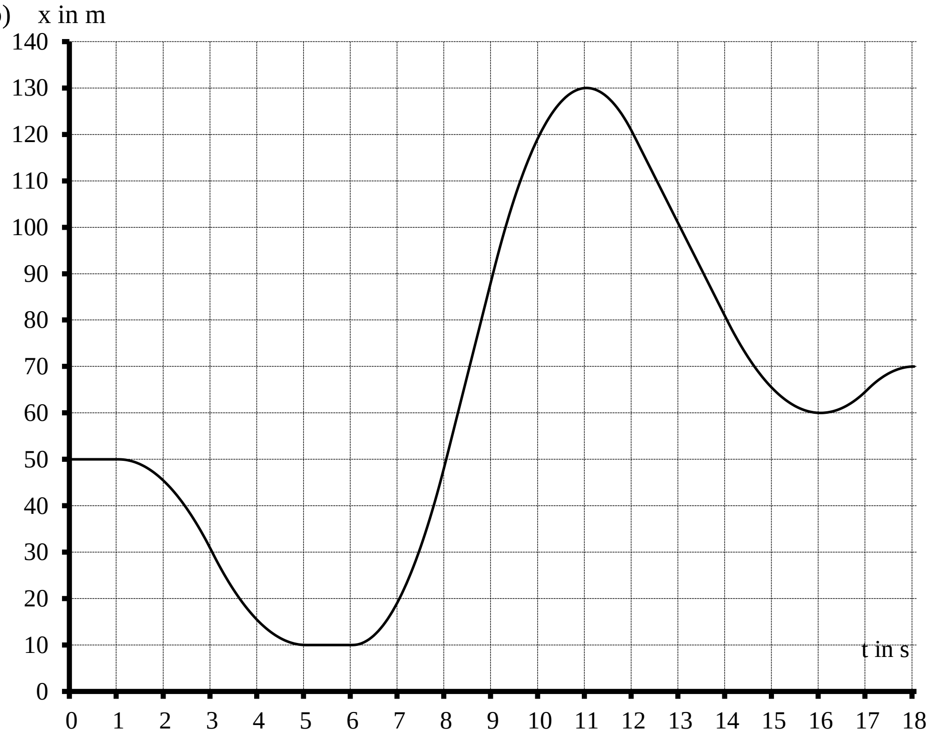
Exercise 2: graphical differentiation
Derive the v-t-diagram
from the x-t-diagram
by graphical differntiation:

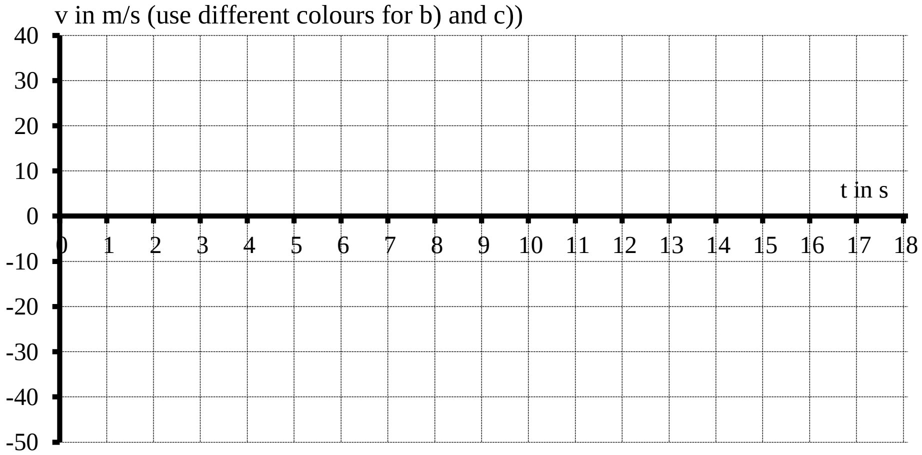
a)



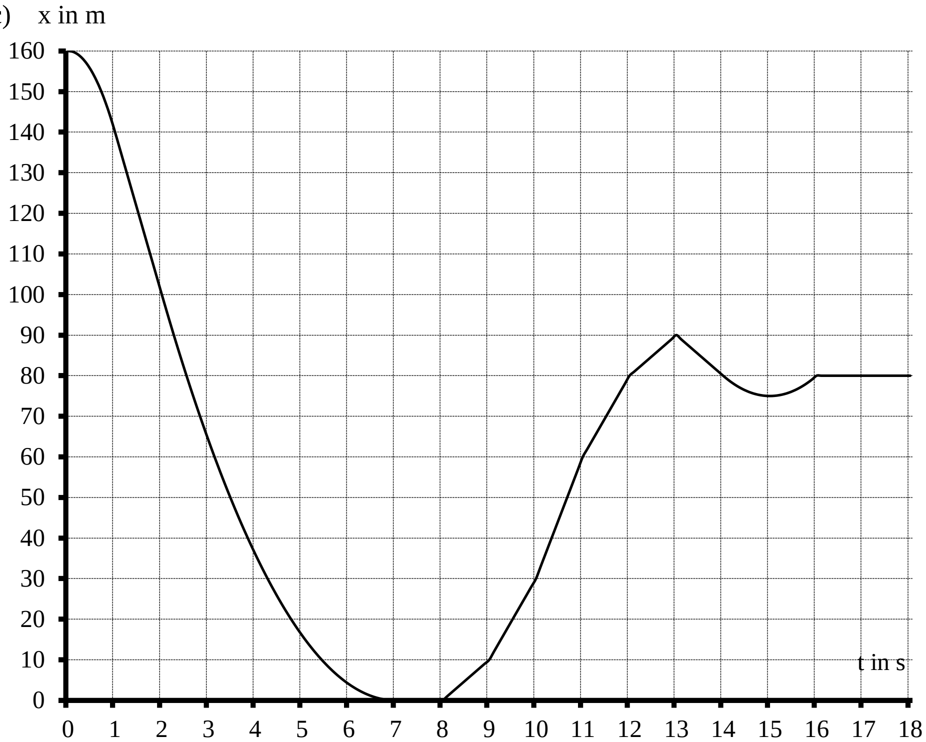


b)

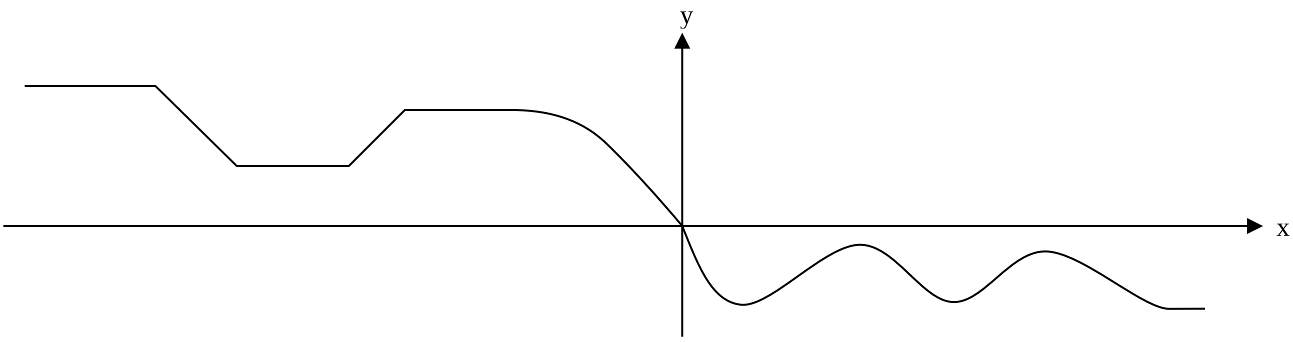

x in m
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
t in s
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18


v in m/s (use different colours for b) and c))
40
30
20
10
0
-10
-20
-30
-40
-50
t in s
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

c)


x in m
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
t in s
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

Exercise 3: Graphical differentiation
Draw the graph of the derivative f'(x).



Exercise 4: Tangent lines
Determine the formula of the tangent line on the curve y = f(x) at the given x.

a) $f(x) = x^2$ for x = 1
b) $f(x) = \frac{1}{2}x^2$ for x = 3
c) $f(x) = x^2 - 1$ for x = −2
d) $f(x) = \frac{1}{2}x^3$ for x = 2
e) $f(x) = x^3 - 1$ for x = −1
f) $f(x) = x^3 - 1$ for x = 1
g) $f(x) = \frac{1}{x}$ for x = 2
h) $f(x) = \frac{1}{x^2}$ for x = 1
i) $f(x) = \sqrt{x}$ for x = 4

Exercise 5: Derivative

Determine the derivative f'(x) for all $x \in D$.
a) $f(x) = x^2$
b) $f(x) = x^3$
c) $f(x) = x^4$
d) $f(x) = x^{-1}$
e) $f(x) = x^{-2}$
f) $f(x) = x^{0,5}$

Exercise 6: Differentiation of power functions
Determine the derivative f'(x) by applying the rule for the differentiation of power functions.

a) $f(x) = x^{33}$
b) $f(x) = x^{\frac{3}{2}}$
c) $f(x) = \sqrt{x}$
d) $f(x) = \frac{1}{x^2}$
e) $f(x) = \sqrt[3]{x}$
f) $f(x) = \frac{1}{\sqrt{x}}$

Exercise 7: Differentiation rules for factors and sums
Determine the derivative f'(x) using difference quotients.

a) $f(x) = \frac{1}{5}x^2$
b) $f(x) = \frac{1}{5}x^3$
c) $f(x) = x^3 + x^2$
d) $f(x) = x^2 - x$

Exercise 8: Differentiation rules for factors and sums
Determine the derivative f'(x) using differentiation rules for power functions, factors and sums.
a) $f(x) = 3x$
b) $f(x) = 3x + 4$
c) $f(x) = 3x^2 - 7x + 54$
d) $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 10$
e) $f(x) = 3(x + 2)^2 + 4$
f) $f(x) = \frac{1}{4}(x - 1)^3$
g) $f(x) = 5 \cdot \sin(x)$
h) $f(x) = 2\cos(x) - 3\sin(x)$

Exercise 9: Intersection angles

The intersection angle is the smaller of the two angles between the two tangents in an intersection point $S \subset f \cap g$ of two curves f and g. Find first the intersection points and the corresponding angles for f and g.
a) $f(x) = x^2 + 2x + 1$ und $g(x) = -x^2 - x$
b) $f(x) = -2x^2 + x$ und $g(x) = -x^3 - x^2$
c) $f(x) = x^3 + 1$ und $g(x) = 2x + 1$
d) $f(x) = x^3 + 3x^2 + 2x + 1$ und $g(x) = x^2 + x + 1$

Exercise 10: Tangent lines with given gradient

Determine all possible contact points $(x_0|y_0)$ on the curve y = f(x) which have a tangent line with gradient a = 2. Also find the formulae of these tangents and draw a graph of the curve with its tangents.

a) $f(x) = x^2 + 2x + 4$
b) $f(x) = \frac{1}{3}x^3 - x^2 - x + \frac{4}{3}$
c) $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 2x + 1$
d) $f(x) = \sqrt{x}$
e) $f(x) = -\frac{1}{2x} + 1$
f) $f(x) = \frac{1}{x^2} - 1$

Exercise 11: Differentiation of trigonometric functions
Determine the formula of the tangent on the curve y = f(x) at the given x by applying the differentiation rule for trigonometric functions.

a) $f(x) = \sin(x)$, $x = \frac{\pi}{4}$ b) $f(x) = \sin(x)$, $x = \frac{\pi}{2}$ c) $f(x) = \cos(x)$, $x = \frac{\pi}{6}$ d) $f(x) = \cos(x)$, $x = \frac{\pi}{3}$

Exercise 12: Tangent lines through given points outside the curve
Determine the formulae of all possible tangents on the curve y = f(x) through the given point P:

a) $f(x) = x^2$ through P(1|−3)
b) $f(x) = x^2 + 4x + 1$ through P(−3|−6)
c) $f(x) = x^2 + 6x + 11$ through P(1|2)
d) $f(x) = x^3 - 4x$ through P(−1|4)
e) $f(x) = x^3 + 6x^2 - 4x - 3$ through P(0|−3)
f) $f(x) = \frac{1}{8}x^3 - x^2$ through P(0|6)

Exercise 13: Tangent lines on curves with parameters.

a) Determine the formulae of all possible tangents on $f(x) = x^2$ through P(u|0) depending on $u \in \mathbb{R}$.
b) For which t has $f_t(x) = -2x^3 + tx^2$ a horizontal tangent line at $x = \frac{1}{3}$?
c) For which t has $f(x) = \frac{1}{3}x^3 - tx^2 + 2x$ **no horizontal tangent at all**?
d) For which t is the tangent line on $f_t(x) = tx^2$ in $P(t|f_t(t))$ **perpendicular** to the tangent line in $Q(-t|f_t(-t))$?

Exercise 14: Envelopes of a family of lines
Show that the following families of lines do not share a common point. Draw their graphs for the given t in a common coordinate system. Guess (!) the formula of the envelope curve f(x). Which conditions must be satisfied so that the points $P(t|g_t(t) = P(t|f(t))$ are contact points? Can you formulate the definition of an envelope curve for a family of lines??

a) $g_t(x) = 2tx - t^2$ with $t \in \mathbb{R}$ for $t \in \{\pm 3; \pm 2; \pm 1; \pm\frac{1}{2}, 0\}$
b) $g_t(x) = -\frac{1}{t^2}x + \frac{2}{t}$ with $t \in \mathbb{R}^*$ for $t \in \{4; 2; 1; \frac{1}{2}, \frac{1}{3}; \frac{1}{4}\}$
c) $g_t(x) = (3t^2 - 4)x - 2t^3$ with $t \in \mathbb{R}$ for $t \in \{\pm 3; \pm 2; \pm 2{,}5; \pm 1; \pm 0{,}5, 0\}$

Exercise 15: Normal lines through given points outside the curve
Determine the formulae of all possible normal lines on the curve y = f(x).
1. through the given **intersection point S on the curve**
2. through the given point p **outside the curve**:

a) $f(x) = x^2$ through S(2|4) resp. through $P(0|\frac{3}{2})$
b) $f(x) = \frac{1}{x}$ through $S(3|\frac{1}{3})$ resp. through $P(\frac{5}{2}|\frac{5}{2})$
c) $f(x) = \frac{1}{x^2}$ through $S(2|\frac{1}{4})$ resp. through $P(0|\frac{1}{2})$
d) $f(x) = \sqrt{x}$ through S(4|2) resp. through $P(\frac{3}{2}|0)$

Exercise 16: Differentiability
Determine the formulae and the domain of the derivative f'(x) and draw a graph of both f and f'.

a) $f(x) = \begin{cases} 1 & \text{for} \quad x < -2 \\ -1 & \text{for} \quad -2 \le x < -1 \\ x & \text{for} \quad -1 \le x < 1 \\ x & \text{for} \quad 1 < x \end{cases}$

b) $f(x) = |x| = \begin{cases} x \text{ for } x \ge 0 \\ -x \text{ for } x < 0 \end{cases}$

c) $f(x) = \operatorname{sgn} x = \begin{cases} 1 \text{ for } x > 0 \\ -1 \text{ for } x < 0 \end{cases}$

d) $f(x) = \begin{cases} -x^2 & \text{for } x \le 0 \\ 0{,}5x^2 & \text{for } x > 0 \end{cases}$

e) $f(x) = \begin{cases} x^2 - 2 & \text{for } x \le 1 \\ 2x - 3 & \text{for } x > 1 \end{cases}$

f) $f(x) = \begin{cases} x^2 + 4x + 4 & \text{for } x \le -1 \\ -x^2 + 2 & \text{for } x > -1 \end{cases}$

5.2. Solutions to the exercises on differentiation

Exercise 1:
Average and instantaneous velocity

a)

$\overline{v_{[0;1]}} = 0\,\frac{m}{s}$ $\quad \overline{v_{[1;3]}} = -1\,\frac{m}{s}$

$\overline{v_{[3;6]}} = -\frac{2}{3}\,\frac{m}{s}$ $\quad \overline{v_{[8;9]}} = 3\,\frac{m}{s}$

$\overline{v_{[10;12]}} = 0\,\frac{m}{s}$ $\quad \overline{v_{[14;16]}} = -1\,\frac{m}{s}$

$\overline{v_{[15;17]}} = 0\,\frac{m}{s}$ $\quad \overline{v_{[16;18]}} = \frac{1}{2}\,\frac{m}{s}$

b)

$v(0) = 0\,\frac{m}{s}$ $\quad v(1) = 0\,\frac{m}{s}$

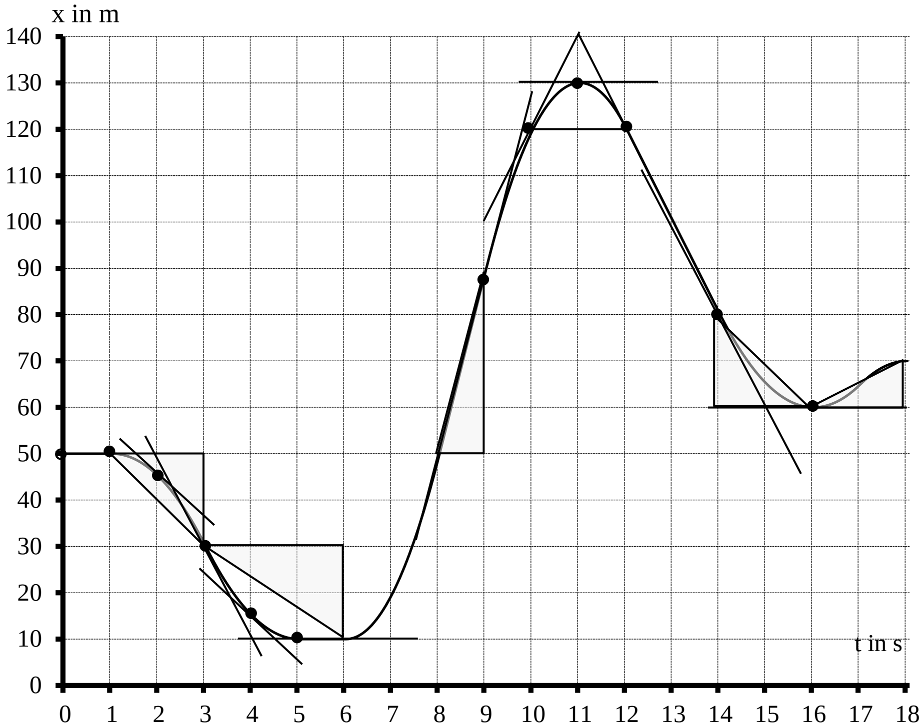
$v(2) = -1\,\frac{m}{s}$ $\quad v(3) = -2\,\frac{m}{s}$

$v(4) = 1\,\frac{m}{s}$ $\quad v(5) = 0\,\frac{m}{s}$

$v(9) = 4\,\frac{m}{s}$ $\quad v(10) = 2\,\frac{m}{s}$

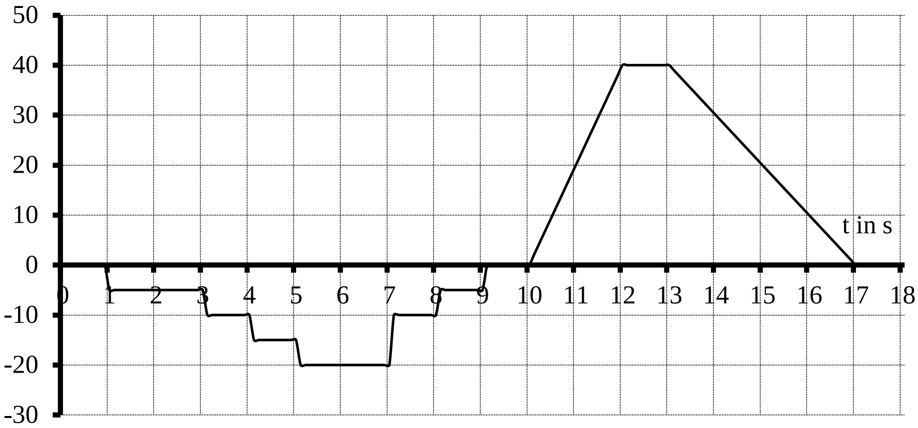
$v(11) = 0\,\frac{m}{s}$ $\quad v(12) = -2\,\frac{m}{s}$

$v(14) = -2\,\frac{m}{s}$ $\quad v(16) = 0\,\frac{m}{s}$

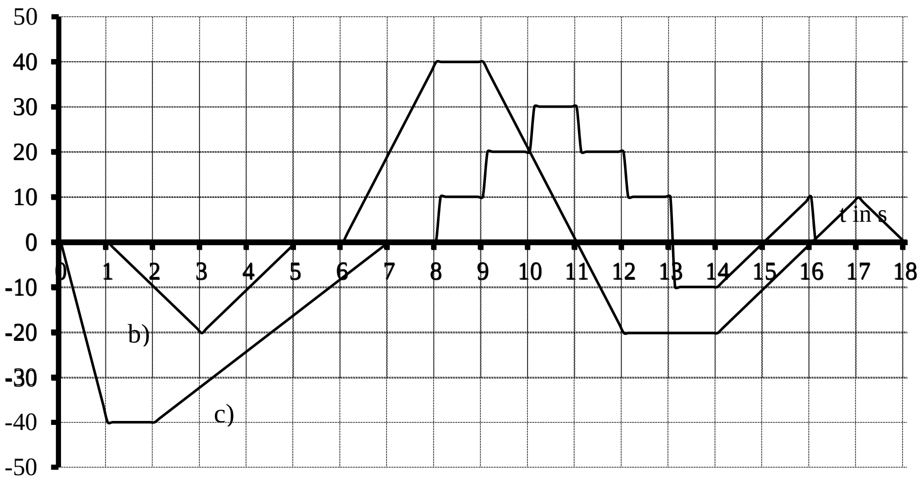


Exercise 2: graphical differentiation

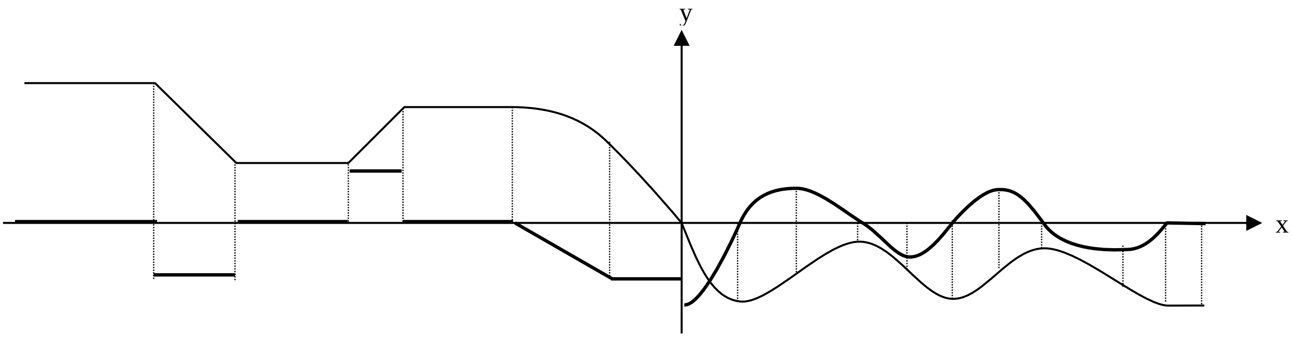
a) v in m/s



b), c) v in m/s



Exercise 3: Graphical differentiation



Exercise 4: Tangent lines

a) $t(x) = 2x - 1$ c) $t(x) = -4x - 5$ e) $t(x) = 3x + 1$ g) $t(x) = -\frac{1}{4}x + 1$ i) $t(x) = \frac{1}{4}x + 1$

b) $t(x) = 3x - \frac{9}{2}$ d) $t(x) = 6x - 8$ f) $t(x) = 3x - 3$ h) $t(x) = -2x + 3$

Exercise 5: Derivatives

a) and b) see script

c) $\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{(x+\Delta x)^4 - x^4}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{x^4+4x^3\cdot\Delta x+6x^2\cdot\Delta x^2+4x\cdot\Delta x^3+\Delta x^4-x^4}{\Delta x} =$

$\lim\limits_{\Delta x\to 0}\frac{\Delta x\cdot(4x^3+6x^2\Delta x+4x\cdot\Delta x^2+\Delta x^3)}{\Delta x} = \lim\limits_{\Delta x\to 0}(4x^3+6x^2\Delta x+4x\cdot\Delta x^2+\Delta x^3) = 4x^3.$

d) $\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{1}{x+\Delta x}-\frac{1}{x}\right) = \lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\cdot\frac{x-x-\Delta x}{x\cdot(x+\Delta x)} = \lim\limits_{\Delta x\to 0}\frac{-1}{x\cdot(x+\Delta x)} = -\frac{1}{x^2}.$

e) $\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}\right) = \lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\cdot\frac{x^2-x^2-2x\cdot\Delta x-\Delta x^2}{(x+\Delta x)^2\cdot x^2} = \lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\cdot\frac{-\Delta x(2x+\Delta x)}{(x+\Delta x)^2\cdot x^2} =$

$\lim\limits_{\Delta x\to 0}\frac{-(2x+\Delta x)}{(x+\Delta x)^2\cdot x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}.$

f) $\lim\limits_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{x+\Delta x-x} = \lim\limits_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\sqrt{x+\Delta x}-\sqrt{x}\ \cdot\ \sqrt{x+\Delta x}+\sqrt{x}} = \lim\limits_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$

Exercise 6: Differentiation of power functions

a) $f'(x) = 33x^{32}$ b) $f'(x) = \frac{3}{2}\sqrt{x}$ c) $f'(x) = \frac{1}{2\sqrt{x}}$ d) $f'(x) = -\frac{2}{x^3}$ e) $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ f) $f'(x) = -\frac{1}{2\sqrt{x^3}}$

Exercise 7: Differentiation of factors and sums

a) $f'(x) = \frac{2}{5}x$ b) $f'(x) = \frac{3}{5}x^2$ c) $f'(x) = 3x^2 + 2x$ d) $f'(x) = 2x - 1$

Exercise 8: Differentiation of factors and sums

a) $f'(x) = 3$ c) $f'(x) = 6x - 7$ e) $f'(x) = 6(x + 2)$ g) $f'(x) = 5\cdot\cos(x)$

b) $f'(x) = 3$ d) $f'(x) = x^3 - x^2 - x + 1$ f) $f(x) = \frac{3}{4}(x - 1)^2$ h) $f'(x) = -2\sin(x) - 3\cos(x)$

Exercise 9: Intersection angles

a) $S_1(-\frac{1}{2}|\frac{1}{4})$ with $\alpha_1 = 45°$ and $S_2(-1|0)$ with $\alpha_2 = 45°$.

b) $S(0|0)$ with $\alpha = 45°$

c) $S_1(0|1)$ with $\alpha_1 = 63{,}44°$ and $S_2(\sqrt{2}\ |2\sqrt{2}\ + 1)$ with $\alpha_2 = 17{,}10°$ and $S_3(-\sqrt{2}\ |-2\sqrt{2}\ + 1)$ with $\alpha_3 = 17{,}10°$

d) $S_1(0|1)$ with $\alpha_1 = 18{,}43°$ and $S_2(-1|1)$ with $\alpha_2 = 0°$ (contact point!)

Exercise 10: Tangent lines with given gradient

a) $t(x) = 2x + 4$ through $P(0|4)$

b) $t_1(x) = 2x + 3$ through $P_1(-1|1)$ and $t_2(x) = 2x - \frac{23}{3}$ through $P_2(3|-\frac{5}{3})$

c) $t_1(x) = 2x + \frac{7}{12}$ through $P_1(-1|-\frac{17}{12})$, $t_2(x) = 2x + 1$ through $P_2(0|1)$ and $t_3(x) = 2x - \frac{5}{3}$ through $P_3(2|\frac{7}{3})$

d) $t(x) = 2x + \frac{1}{8}$ through $P(\frac{1}{16} | \frac{1}{4})$

e) $t(x) = 2x + 3$ through $P(-\frac{1}{2} | 2)$ and $t(x) = 2x - 1$ through $P(\frac{1}{2} | 0)$

f) $t(x) = 2x + 2$ through $P(-1 | 0)$

Exercise 11: Differentiation of trigonometric functions

a) $t(x) = \frac{1}{\sqrt{2}}x - \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$ b) $t(x) = 1$ c) $t(x) = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$ d) $t(x) = -\frac{\sqrt{3}}{2}x - \frac{\pi}{2\sqrt{3}} + \frac{1}{2}$

Exercise 12: Tangent lines through given points outside the curve

a) $t_1(x) = -2x - 1$ with contact point $B(-1|1)$ and $t_2(x) = 6x - 9$ with contact point $B(3|9)$.

b) $t_1(x) = 2x$ with contact point $B(-1|-2)$ and $t_2(x) = -6x - 24$ with contact point $B(-5|6)$.

c) $t_1(x) = 16x - 14$ with contact point $B(5|66)$ and $t_2(x) = 2$ with contact point $B(-3|2)$

d) $t_1(x) = -4x$ with contact point $B(0|0)$ and $t_2(x) = \frac{11}{4}x + \frac{27}{4}$ with contact point $B(-\frac{3}{2} | \frac{21}{8})$.

e) $t_1(x) = -4x - 3$ with contact point $B(0|-3)$ and $t_2(x) = -13x - 3$ with contact point $B(-3|36)$

f) $t(x) = \frac{11}{2}x + 6$ with contact point $B(-2|-5)$

Exercise 13: Tangent lines including parameters

a) $t_1(x) = 0$ with contact point $B(0|0)$ und $t_{u2}(x) = 4ux - 4u^2$ with contact point $B(2u|4u^2)$

b) The derivative $f_t'(x) = -6x^2 + 2tx$ must vanish at $x = \frac{1}{3}$: $0 = f_t'(\frac{1}{3}) = -\frac{2}{3} + \frac{2}{3}t \Rightarrow t = 1$.

c) $0 = f_t'(x) = x^2 - 2tx + 2 \Rightarrow x = t \pm \sqrt{t^2 - 2} \Rightarrow$ **no** solution for $-\sqrt{2} < t < \sqrt{2}$

d) The gradient of the first tangent line is $2t \cdot t = 2a^2$ and that of the second one is $2t \cdot (-t) = -2t^2$. The tangent lines a perpendicular if $2t^2 = -\frac{1}{-2t^2}$, i.e. for $t = \pm \frac{1}{\sqrt{2}}$.

Exercise 14: Envelope of a family of lines

Each member of the family $\{g_t: t \in \mathbb{R}\}$ is a tangent line of the envelope f with contact point $P(t|f(t))$ so that for $x = t$ the y-values as well as the gradients must be equal: $f(t) = g_t(t)$ und $f'(t) = g_t'(t)$. These two conditions result in the formulae:

a) $f(x) = x^2$ b) $f(x) = \frac{1}{x}$ c) $f(x) = x^3 - 4x$

Exercise 15: Normal lines through given points

a) $n_0(x) = -\frac{1}{4}x + \frac{9}{2}$ through $S(2|4)$, $n_{1/2}(x) = \pm\frac{1}{2}x + \frac{3}{2}$ through $S_{1/2}(\pm1|1)$ und n_3: $x = 0$ through $S_3(0|0)$

b) $n_0(x) = 9x - \frac{80}{3}$ with $S(3|\frac{1}{3})$, $n_1(x) = \frac{1}{4}x + \frac{15}{8}$ with $S_1(\frac{1}{2}|2)$, $n_2(x) = x$ with $S_2(\pm1|\pm1)$, $n_3(x) = 4x - \frac{15}{2}$ with $S_3(2|\frac{1}{2})$

c) $n_0(x) = 4x - \frac{31}{4}$ through $S(2|\frac{1}{4})$ and $n_{1/2}(x) = \pm\frac{1}{2}x + \frac{1}{2}$ through $S_{1/2}(\pm1|1)$

d) $n_0(x) = -4x + 18$ through $S(4|2)$ and $n_1(x) = -2x + 3$ through $S_1(1|1)$ ($n_2(x) = 0$ through $S(0|0)$ isn't a normal line because the sqareroot doesn't have a (both-sided) derivative in $S(0|0)$!)

Exercise 16: Differentiability

a) see script

b) $f'(x) = \operatorname{sgn}(x)$ with $D = \mathbb{R}\backslash\{0\}$

c) $f'(x) = 0$ with $D = \mathbb{R}\backslash\{0\}$

d) $f'(x) = \begin{cases} -2x & \text{for } x \le 0 \\ x & \text{for } x > 0 \end{cases}$ with $D = \mathbb{R}$

e) $f'(x) = \begin{cases} 2x & \text{for } x \le 1 \\ 2 & \text{for } x > 1 \end{cases}$ with $D = \mathbb{R}$

f) $f'(x) = \begin{cases} 2x + 4 & \text{for } x \le -1 \\ -2x & \text{for } x > -1 \end{cases}$ with $D = \mathbb{R}$