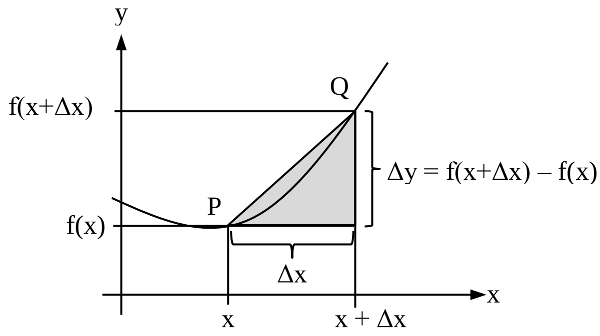
5.2. Differentiation

5.2.1. The average gradient of a function between two points

The **average gradient** of the function f(x) between the two points P(x|f(x)) and Q(x + Δx|f(x + Δx))



is defined as the **difference quotient**

$$\frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

It can be interpreted as

- the **slope** of f (How **steep** is f?), if x means **place** or
- its **rate of change** (How **fast** changes f?), if x means **time**,

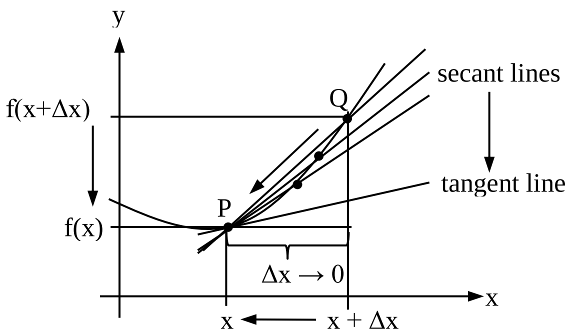
especially

the **average velocity v** = $\frac{\Delta x}{\Delta t} = \frac{\text{change in place}}{\text{change in time}}$

Exercises on differentiation No. 1a

5.2.2. The instantaneous gradient of a function at a point

The **instantaneous gradient** or **derivative** f'(x) (pronounced "f prime") of a function y = f(x) at a point P(x|f(x)) is defined as the gradient of the **tangent line** in P.



Graphically this tangent line can be approximated by **secant lines** through points P(x|f(x) and Q(x + Δx)|f(x + Δx)) with Q moving ever closer to P.

Algebraically the gradient of the tangent is the **limit of the gradients of the secant lines** for Δx → 0:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative can also be expressed in **differential** notation:

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{or} \quad f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

It is essential to be familiar with these notations since all of them are used and a mathematical text may very well switch between notations depending on circumstances:

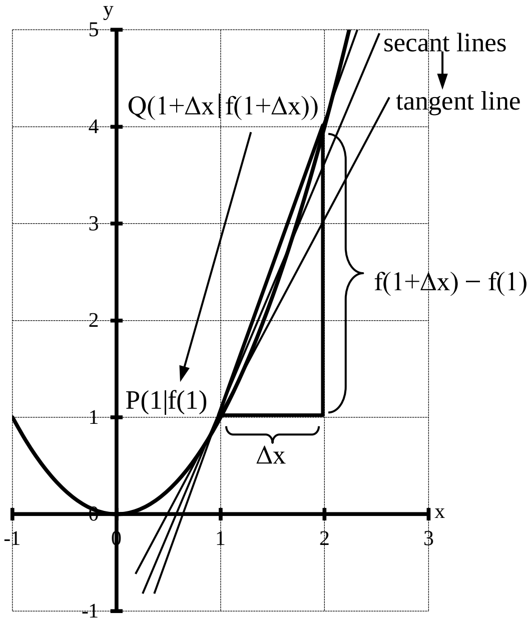
average gradient	= gradient of the **secant** line	= difference quotient =	$\frac{\Delta y}{\Delta x}$ =	$\frac{\Delta f}{\Delta x}$ =	$\frac{f(x+\Delta x) - f(x)}{\Delta x}$
↓ Δx → 0	↓ Δx → 0	↓ Δx → 0	↓ Δx → 0	↓ Δx → 0	↓ Δx → 0
instantaneous gradient	= gradient of the **tangent** line =	differential =	$\frac{dy}{dx}$ =	$\frac{df}{dx}$ =	f'(x)

Exercises on differentiation No. 1b, 2 and 3

Example: Gradient of the parabola f(x) = x^2 when x = 1

The instantaneous gradient (slope, rate of change) or **derivative** f'(1) of the parabola f(x) = x^2 when x = 1 can be determined **geometrically** by drawing the **tangent line** through the point P(1|f(1)) = P(1|1) and measuring its gradient. **Algebraically** the gradient of the tangent line is the **limit** of the gradients of the secant lines through P(1|1) and Q(1+Δx|f(1+Δx)) for Δx → 0:

$$
\begin{aligned}
f'(1) &= \lim_{\Delta x\to 0}\frac{f(1+\Delta x)-f(1)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(1+\Delta x)^2-1^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\cancel{1^2}+2\Delta x+(\Delta x)^2-\cancel{1^2}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2\Delta x+(\Delta x)^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\cancel{\Delta x}(2+\Delta x)}{\cancel{\Delta x}}\\
&= \lim_{\Delta x\to 0}(2+\Delta x)\\
&= 2+0\\
&= 2
\end{aligned}
$$



So the tangent line has the formula t(x) = 2x + b.

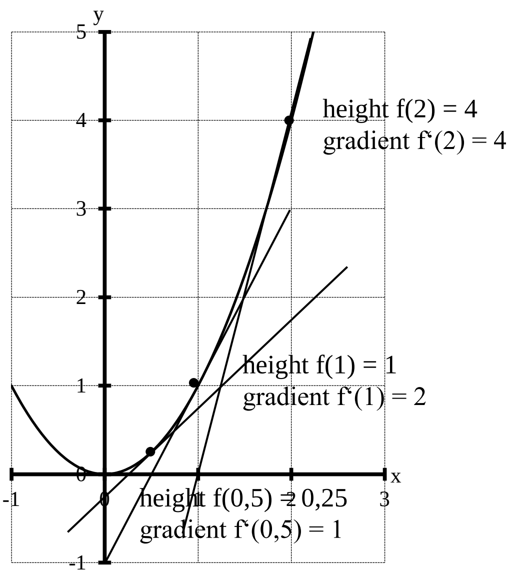
When we plug in the coordinates of the touching point P(1|1)
we get the equation 1 = 2·1 + b and thereby the y-intercept b = −1.

The tangent line is t(x) = 2x − 1.

Exercises on Differentiation No. 4

Example: Derivative of the parabola f(x) = x^2 at some x ∈ ℝ:

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\cancel{x^2}+2x\cdot\Delta x+(\Delta x)^2-\cancel{x^2}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2x\cdot\Delta x+(\Delta x)^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}}\\
&= \lim_{\Delta x\to 0}(2x+\Delta x)\\
&= 2x+0\\
&= 2x
\end{aligned}
$$



The tangent line on the parabola f(x) = x^2 through the point P(x|f(x)) has the gradient f'(x) = 2x.

In other words: the derivative of f(x) = x^2 is f'(x) = 2x.

Example: Derivative of the cubic parabola f(x) = x^3 at some x ∈ ℝ

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{x^3+3x^2\cdot\Delta x+3x\cdot(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{3x^2\cdot\Delta x+3x\cdot(\Delta x)^2+(\Delta x)^3}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{\Delta x(3x^2+3x\cdot\Delta x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}(3x^2+3x\cdot\Delta x)$$
$$= 3x^2+0$$
$$= 3x^2.$$

Exercises on differentiation No. 5

5.2.3. Derivatives of polynomial functions

5.2.3.1. Power functions

Theorem: Derivative of the power functions

$(x^n)' = nx^{n-1}$ for n ∈ ℝ

Proof for n ∈ ℕ:

$$(x^n)' = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{\cancel{x^n}+n\cdot x^{n-1}\cdot\Delta x+\ldots+n\cdot x\cdot(\Delta x)^{n-1}+(\Delta x)^n-\cancel{x^n}}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{\cancel{\Delta x}\cdot\ n\cdot x^{n-1}+\ldots+n\cdot x\cdot(\Delta x)^{n-2}+(\Delta x)^{n-1}}{\cancel{\Delta x}}$$
$$= \lim_{\Delta x\to 0} n\cdot x^{n-1}+\ldots+n\cdot x\cdot(\Delta x)^{n-2}+(\Delta x)^{n-1}$$
$$= n\cdot x^{n-1}+\ldots+0+0$$
$$= n\cdot x^{n-1}, \text{ qed}$$

Examples:

f(x)	f'(x)
x^0	0
x^1	$1x^0$
x^2	$2x^1$
x^3	$3x^2$
x^4	$4x^3$
x^n	nx^{n-1}

Exercises on differentiation No 6

5.2.3.2. Constant factors

Example: Determine the derivative of f(x) = $\frac{1}{2}x^3$ at some x ∈ ℝ.

Solution: $$\left(\frac{1}{2}x^3\right)' = \lim_{\Delta x\to 0}\frac{\frac{1}{2}(x+\Delta x)^3-\frac{1}{2}x^3}{\Delta x} = \frac{1}{2}\cdot\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x} = \frac{1}{2}3x^2 = \frac{1}{2}(x^3)'.$$

Theorem: Constant factors remain unchanged: $(a\cdot f(x))' = a\cdot f'(x)$:

Proof: $$(a\cdot f(x))' = \lim_{\Delta x\to 0}\frac{a\cdot f(x+\Delta x)-a\cdot f(x)}{\Delta x} = a\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = a\cdot f'(x) = (a\cdot f(x))'$$

5.2.3.3. Sum rule

Example: Determine the derivative of $f(x) = x^3 + x^2$ at some $x \in \mathbb{R}$.

Solution:

$$(x^3 + x^2)' = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^3+(x+\Delta x)^3-(x^3+x^2)}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{(x+\Delta x)^3-x^3}{\Delta x}+\frac{(x+\Delta x)^2-x^2}{\Delta x}\right) = 3x^2 + 2x = (x^3)' + (x^2)'$$

Theorem: The derivative of a sum is the sum of the derivatives: $(f(x) + g(x))' = f'(x) + g'(x)$.
Proof:

$$(f(x) + g(x))' = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right) = f'(x) + g'(x).$$

Exercises on differentiation No 7

Theorem: Derivative of polynomial functions
$(a_nx^n + ... + a_2x^2 + a_1x + a_0)' = na_nx^{n-1} + ... + 2a_1x + a_1$ for $n \in \mathbb{N}$ and $a_n, ..., a_0 \in \mathbb{R}$ with $a_n \neq 0$.
Proof:
Apply the three preceding theorems about power functions, constant factors and sums of functions.

Exercises on differentiation No 8 - 10

5.2.4. Derivatives of the trigonometric functions

Theorem: sin'(x) = cos(x) and cos'(x) = −sin(x)
Proof: The **limits** $\lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x} = 1$ and $\lim_{\Delta x\to 0}\frac{\cos(\Delta x)-1}{\Delta x} = 0$ are needed as well as the **addition theorems** $\sin(\alpha + \beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta)$ and $\cos(\alpha + \beta) = \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)$:

$$\begin{aligned}\sin'(x) &= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(\frac{\sin(x)\cos(\Delta x)}{\Delta x}+\frac{\cos(x)\sin(\Delta x)}{\Delta x}-\frac{\sin(x)}{\Delta x}\right)\\ &= \lim_{\Delta x\to 0}\left(\sin(x)\frac{\cos(\Delta x)-1}{\Delta x}+\cos(x)\frac{\sin(\Delta x)}{\Delta x}\right)\\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1\\ &= \cos(x).\end{aligned}$$

und

$$\begin{aligned}\cos'(x) &= \lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(\frac{\cos(x)\cos(\Delta x)}{\Delta x}-\frac{\sin(x)\sin(\Delta x)}{\Delta x}-\frac{\cos(x)}{\Delta x}\right)\\ &= \lim_{\Delta x\to 0}\left(\cos(x)\frac{\cos(\Delta x)-1}{\Delta x}-\sin(x)\frac{\sin(\Delta x)}{\Delta x}\right)\\ &= \cos(x)\cdot 0 - \sin(x)\cdot 1\\ &= -\sin(x).\end{aligned}$$

Exercises on differentiation No 11

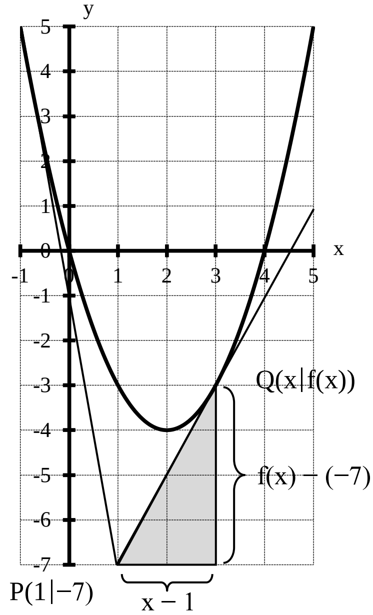
5.2.5. Tangent and normal lines

Example for determining a tangent line on a curve through a given point outside the curve:

Find the formulae of all possible tangent lines through P(1|−7) on $f(x) = x^2 - 4x$.

Solution:

First we seek the **x-coordinates of the contact points** with the approach



$$\text{gradient} = \text{derivative}$$

$$\frac{f(x)-(-7)}{x-1} = f'(x)$$

$$\frac{x^2-4x-(-7)}{x-1} = 2x - 4 \qquad | \cdot(x-1)$$

$$x^2 - 4x + 7 = 2x^2 - 6x + 4 \qquad | -x^2 + 4x - 7$$

$$0 = x^2 - 2x - 3 \qquad | \text{ solving the quadratic equation}$$

$\Rightarrow$ $x_1 = -1$ and $x_2 = 3$

The **y-coordinates of the contact points** are the values of $f(x) = x^2 - 4x$:
$y_1 = f(x_1) = 5$ and $y_2 = f(x_2) = -3$

The **gradients a of the tangent formula y = ax + b** are the values of $f'(x) = 2x - 4$:
$a_1 = f'(x_1) = -6$ and $a_2 = f'(x_2) = 2$

The **y-intercept b of the tangent formula y = ax + b** can be obtained by substituting the **coordinates of the given point** in y = ax + b:
$b_1 = y - a_1x = -7 - (-6)\cdot 1 = -1$ and $b_2 = y - a_2x = -7 + 2\cdot 1 = -5$

Hence there are two tangent lines $t_1(x) = -6x - 1$ with contact point $B_1(-1|5)$ and $t_2(x) = 2x - 5$ with contact point $B_2(3|-3)$

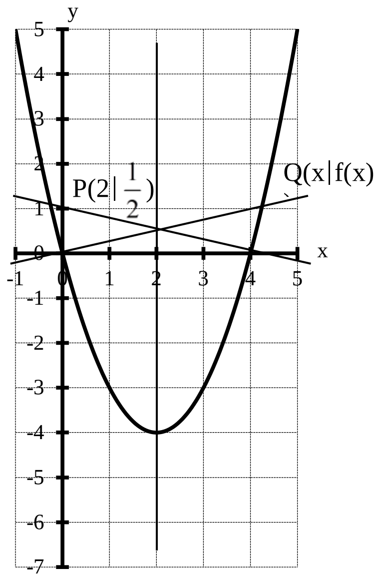
Exercises on differentiation No 12 - 14

Example for determining a normal line on a curve through a given point outside the curve:

Find the formulae of all possible normal lines through $P(2|\frac{1}{2})$ on $f(x) = x^2 - 4x$.

Solution:

Determine the **x-coordinates of the contact points** with the approach



$$\text{gradient} = -\frac{1}{\text{derivative}}$$

$$\frac{f(x)-\frac{1}{2}}{x-2} = -\frac{1}{f'(x)}$$

$$\frac{x^2-4x-\frac{1}{2}}{x-2} = -\frac{1}{2x-4} \qquad | \cdot(x-2); \cdot(2x-4)$$

$$(x^2 - 4x - \frac{1}{2})(2x-4) = -(x-2) \qquad | \text{ expand}$$

$$2x^3 - 12x^2 + 15x + 2 = -x + 2 \qquad | + x - 2$$

$$2x^3 - 12x^2 + 16x = 0 \qquad | \text{ factorize } 2x$$

$$2x(x - 6x + 8) = 0 \qquad | \text{ solve quadratic equation}$$

$\Rightarrow$ $x_1 = 0$, $x_2 = 2$ and $x_3 = 4$

The **y-coordinates of the intersection points** are the values of $f(x) = x^2 - 4x$:
$y_1 = f(x_1) = 0$, $y_2 = f(x_2) = -4$ und $y_3 = f(x_3) = 0$

The **gradients of the normal lines** are $a = -\frac{1}{f'(x)} = -\frac{1}{2x-4}$

$a_1 = -\frac{1}{f'(x_1)} = \frac{1}{4}$, a_2 is not defined because the normal line is vertical and $a_3 = -\frac{1}{f'(x_2)} = -\frac{1}{4}$

The **y-intercepts b of the normal lines** can be obtained by substituting the **coordinates of the given point** in y = ax + b

$b_1 = y - a_1x = \frac{1}{2} - \frac{1}{4}\cdot 2 = 0$ und $b_3 = y - a_3x = \frac{1}{2} - (-\frac{1}{4})\cdot 2 = 1$

Hence there are **three normal lines**:

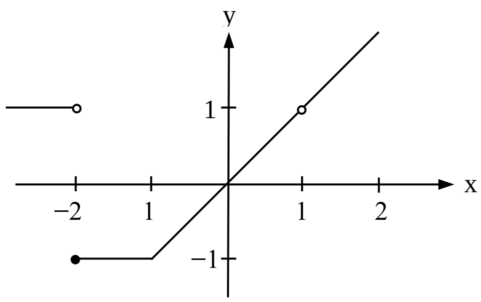
1. $n_1(x) = \frac{1}{4}x$ with intersection point $S_1(0|0)$,
2. n_2: x = 2 with intersection point $S_2(2|-4)$ and
3. $n_3(x) = -\frac{1}{4}x + 1$ with intersection point $S_3(4|0)$

Exercises on differentiation No 15

5.2.6. Differentiability und continuity at x

Example: Exercises on differentiation No 16a

$$f(x) = \begin{cases} 1 & \text{für} \quad x < -2 \\ -1 & \text{für} \quad -2 \le x < -1 \\ x & \text{für} \quad -1 \le x < 1 \\ x & \text{für} \quad 1 < x \end{cases}$$



The derivative f'(x) on the straight lines is obvious:

$$f'(x) = \begin{cases} 0 & \text{für} \quad x < -2 \\ 0 & \text{für} \quad -2 < x < -1 \\ 1 & \text{für} \quad -1 < x < 1 \\ 1 & \text{für} \quad 1 < x \end{cases}$$

More interesting are the derivatives at the **joints**

- the **jump** at x = −2,
- the **knee** at x = −1 und
- the **hole** at x = 1.

jump at x = −2:

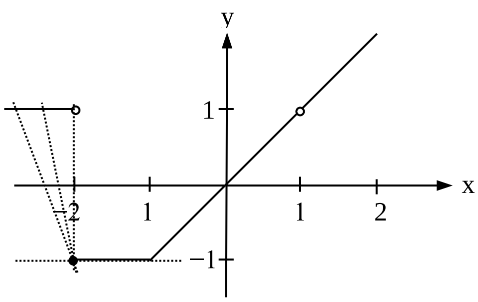
The derivative is defined as the limit of the difference quotient

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \begin{cases} \frac{-1-(-1)}{\Delta x} = 0 & \text{for } \Delta x > 0 \\ \frac{1-(-1)}{\Delta x} = \frac{2}{\Delta x} & \text{for } \Delta x < 0 \end{cases} \quad \text{for } \Delta x \to 0$$

Δx	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$
−0,1	−20
−0,01	−200
−0,001	−2000
−0	-
+0,001	0
+0,01	0
+0,1	0

Unfortunately in this case there are two different limits depending on the direction you come from:
When approaching from the **right side** ($\Delta x > 0$), the difference quotient is constantly 0.
But coming from the **left side** ($\Delta x < 0$) the difference quotient tends towards $-\infty$:

Correspondingly the secant lines on the right "bank" are horizontal while those connecting with the left bank get steeper and steeper



Thus it is not possible to determine a single gradient at x = −2:
f is **not differentiable** at x = −2

Knee at x = −1

The difference quotient at x = −1 is

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \begin{cases} \frac{x-(-1)}{\Delta x} = 1 \text{ für } \Delta x > 0 \\ \frac{-1-(-1)}{\Delta x} = 0 \text{ für } \Delta x < 0 \end{cases} \text{ für } \Delta x \to 0$$

Again there are two different limits depending in where you come from: While the secants on the left side are horizontal with gradient 0 those on the right side have the gradient 1. Since there is no definite limit f is not differentiable at x = −1.

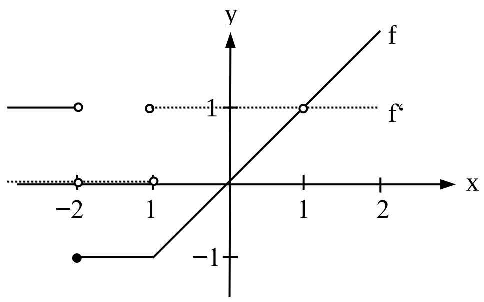
Hole at x = 1

Since there is no f(1) the gradient $\frac{f(1+\Delta x)-f(1)}{\Delta x}$ cannot be evaluated and consequently f is not differentiable at x = 1 either.

Summary

Every knee, jump or hole in f produces a hole in f':

$$f(x) = \begin{cases} 1 & \text{for} \quad x < -2 \\ -1 & \text{for} \quad -2 \le x < -1 \\ x & \text{for} \quad -1 \le x < 1 \\ x & \text{for} \quad 1 < x \end{cases} \quad \text{with } D = \mathbb{R}\backslash\{1\}$$



$$f'(x) = \begin{cases} 0 & \text{for} \quad x < -2 \\ 0 & \text{for} \quad -2 < x < -1 \\ 1 & \text{for} \quad -1 < x < 1 \\ 1 & \text{for} \quad 1 < x \end{cases} \quad \text{with } D = \mathbb{R}\backslash\{-2, -1, 1\}$$

Differentiability and continuity

- The function f is **differentiable** at x, if $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ = f'(x) exists. Its graph is **smooth** without knees.

  Functions are **not differentiable** at **jumps**, **knees**, and **holes**.
- The function f is **continuous** at x, if $\lim\limits_{\Delta x \to 0} f(x+\Delta x)$ = f(x). Its graph has **no interruptions**.

  Functions are **not continuous** at **jumps** and **holes**.
- **If f is differentiable then** $\lim\limits_{\Delta x \to 0} f(x+\Delta x)-f(x)$ = 0 and so must be **continuous too.**

Exercises on differentiation No 16 b - f