5.3. Exercises on the curve analysis of polynomial functions

Exercise 1: Curve analysis
Examine the following functions on symmetry, x- and y-intercepts, extrema and inflexion points. Draw their graphs including all significant points on the domain given.

Instruction on how to determine extrema
1. Mark **increasing** sections with a **continuous red** line and **decreasing** sections with a **dotted red** line.
2. Find the **first derivative** f' and its zeroes. Draw the graph of f' as a **red line** into the same coordinate system.
3. Formulate a **rule** on how to find maxima and minima of a given function f using the **sign changes of the first derivative**.

Instruction on how to determine inflexion points
1. Mark **convex** (increasing slope/turning left) sections with a **continuous green** line and **concave** (decreasing slope/turning right) sections with a **dotted green** line.
2. Find the **second derivative f''** and its zeroes. Draw the graph of f'' as a **green line** into the same coordinate system.
3. Formulate a rule on how to find **inflexion points** (where the changes from convex to concave or vice versa) of a given function f using the **extrema of the first derivative f'** or the **sign changes of the second derivative f''**.

a) $f(x) = \frac{1}{3}x^3 - \frac{4}{3}x$

b) $f(x) = \frac{1}{4}x^4 - x^2$

c) $f(x) = \frac{1}{3}(x^3 - 1)$

d) $f(x) = \frac{1}{4}(x^4 - 1)$

e) $f(x) = \frac{1}{5}(x^5 - 1)$

f) $f(x) = -\frac{1}{3}x^3 + 3x$

g) $f(x) = \frac{1}{6}x^3 - \frac{3}{2}x^2 + 4x$

h) $f(x) = -\frac{1}{4}x^3 + x^2$

i) $f(x) = \frac{1}{12}x^4 - x^2 + 6$

j) $f(x) = -x^4 + 5x^2 - 4$

k) $f(x) = -\frac{1}{4}x^4 + 2x^2 + \frac{9}{4}$

l) $f(x) = -\frac{1}{20}x^5 + \frac{5}{6}x^3 - \frac{9}{4}x$ (no intercepts asked!)

m) $f(x) = \frac{1}{4}x^5 - \frac{25}{12}x^3 + 5x$

n) $f(x) = \frac{1}{10}(x + 1)(x - 2)^3$ (polynomial long division needed!)

Exercise 2: Common points of families of functions
Draw the graphs for the t given into a single coordinate system. Determine graphically the common point $\mathbf{P(x_0|y_0)}$ of all curves and prove your conjecture using the following criterion: $P(x_0|y_0)$ is a common point if for any t_1, t_2 the pair f_{t1} and f_{t2} meet in P, that is: $f_{t1}(x_0) = f_{t2}(x_0) = y_0$.

a) $f_t(x) = x^2 + 2tx + 1 - 2t$ for t = 0, ± 1 and ± 2

b) $f_t(x) = \frac{1}{x - t} + \frac{t}{t - 1}$ for t = −1, 0, 2 and 3

c) $f_t(x) = x^3 + 3x^2 + (3 - t^2)x + 1 + t^2$ for t = 0, ± 1 and ± 2

Exercise 3: Envelopes of families of lines
Draw the graphs for the t given into a single coordinate system. Determine graphically the envelope f of the line family and prove your conjecture using the following criterion: The curve f(x) is an envelope of the family g_t, if the lines g_t are tangents to f at x = t, i.e. if $f(t) = g_t(t)$ and $f'(t) = g_t'(t)$ for any t.

a) $g_t(x) = 2tx - t^2$ for $t = 0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \pm 2$ and ± 3

b) $g_t(x) = -\frac{1}{t^2}x + \frac{2}{t}$ for $t = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3$ and 4

c) $g_t(x) = \frac{1}{2\sqrt{t}}x + \frac{\sqrt{t}}{2}$ for $t = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3$ and 4

Exercise 4: Loci of extrema and inflexion points

Determine the coordinates of all extrema and inflexion points as well as the formulae of their loci. The locus of a Point (x(t)|y(t)) is the curve y = y(x) which results if the parameter t is eliminated from the two equations x = x(t) and y = y(t).

a) $f_t(x) = x(x - t)$ with $t \in \mathbb{R}$

b) $f_a(x) = x^2 + 6x + a$ with $a \in \mathbb{R}$

c) $f_a(x) = x^2 + ax + 2$ with $a \in \mathbb{R}$

d) $f_a(x) = ax^2 - 8x - 5$ with $a \in \mathbb{R}$

e) $f_t(x) = -\frac{t^3}{16}x^2 + \frac{t^2}{2}x$ with $t \in \mathbb{R}_+^*$

f) $f_k(x) = x^3 + \frac{6-3k}{2}x^2 - 6kx + 2$ with $k \in \mathbb{R}_+^*$

g) $f_a(x) = \frac{1}{a^2}x^3 - \frac{3}{a}x$ with $a \in \mathbb{R}_+^*$

h) $f_t(x) = \frac{1}{6t}x^3 - x^2 + \frac{3}{2}tx$ with $t \in \mathbb{R}_+^*$

i) $f_t(x) = ax^3 - x^2$ with $a \in \mathbb{R}^*$

j) $f_b(x) = x^4 + bx^3$ with $b \in \mathbb{R}_+^*$

k) $f_t(x) = -t^2x^3 + 2x + 1$ with $t > 0$

l) $f_t(x) = \frac{1}{x} - \frac{t}{x^2}$ with $t \in \mathbb{R}$

Exercise 5: Finding formulae of functions with given properties

Determine the formulae of the functions with the following properties:

a) f is a polynomial of degree 3 and point symmetric with regard to the origin O(0|0). In O it also has the tangent y = 2x and it intercepts the x-axis at $x_1 = 2$.

b) f is a polynomial of degree 3 with extremum E(−2|−2) and inflexion point W(0|−4).

c) f is a polynomial of degree 3 with horizontal tangent at x = 0. The tangent line through the inflexion point W(2|2) has the formula y = x.

d) f is a polynomial of degree 3 with x-intercept N(4|0), relative maximum H(2|7) and relative minimum $T(1|\frac{27}{4})$.

e) f is a polynomial of degree 3 with inflexion point $W(1|\frac{5}{4})$. It intersects the parabola $g(x) = \frac{1}{4}x^2 - x + 2$ at $x_0 = -1$ and has gradient 1 in this intersection point.

f) f is a polynomial of degree 4 and its graph is symmetric with respect to the y-axis. It has an inflexion point W(1|2) and the tangent line in this point also passes through the origin O(0|0).

g) f is a polynomial of degree 4 and its graph has the inflexion point W(2|0) with gradient 0. The normal line through the origin O(0|0) has the gradient $-\frac{1}{8}$.

h) f is a polynomial of degree 5 and its graph is point symmetric with respect to the origin O(0|0). It has a horizontal tangent line through the inflexion point W(2|f(2)) with y-intercept $y = \frac{64}{25}$.

i) f_t is a family of cubic parabolas with common point O(0|0) and gradient $\frac{3}{2}t$ in this point. They touch the x-axis in N(3t|0).

j) f_t is a family of cubic parabolas which are point symmetric with respect to the origin and common point P(2|4).

Exercise 6: Optimization problems

a) The vertical line x = u moves in the domain $0 \le u \le 1$. It intersects the x-axis in P and the parabola $f(x) = -\frac{1}{4}(x^2 - 2x - 15)$ in Q. The x-intercept of the parabola with positive x-value is R. Which u results in the triangle PQR with maximal area?

b) The vertical line x = u intersects $f(x) = (x - 2)^2 - 2$ in B and $g(x) = (x - 4)^2 + 2$ in C. Which u in the domain $1 \le u \le 4$ yields the maximal area for the triangle ABC with additional vertex A(1|2)?

Exercise 7: Optimization problems with respect to distance and space between two graphs

The vertical line x = u can move through the whole area between the graphs of f and g. P and Q are the intersection points with f and g respectively. O(0|0) is the origin.

1. Which u results in the maximal distance $\overline{PQ}$?
2. Which u gives the maximal area for the triangle OPQ?

a) $f(x) = x^2 - 6x$ und $g(x) = -x^2$

b) $f(x) = x^2$ und $g(x) = x(x - 2)^2$

Exercise 8: Optimization problems without given coordinates

a) Find the vertices of the rectangle with maximal area between the parabola $f(x) = -x^2 + 4$ and the coordinate axes.

b) Find the vertices of the rectangle with maximal area between $f(x) = x^4 - 5x^2 - 20$. and the x-axis.

c) Find the vertices B and C of the triangle ABC with $A(-1|1)$ between the parabola $f(x) = x^2 - 3$ and the x-axis with maximal area and horizontal base line BC.

d) Fin the vertices of the rectangle with horizontal base between $f(x) = 3 - \frac{2}{3}x^2$ and $g(x) = \frac{1}{3}x^2 - 6$.

Exercise 9: Optimization problem with shift

a) Show that $f(x) = -x^2 + 4x - 3$ is symmetrical with respect to the axis $x = 2$. Find the triangle ABC with maximal area, horizontal base line AB and vertex C on the x-axis.

b) Find the vertices of the rectangle with maximal area between the parabola $f(x) = -x^2 + 4x$ and the coordinate axes.

Exercise 10: Optimization problem with curve family

Given is the family $f_t(x) = x^2(x - t)$. Find the u_t, so that the right angled triangle with vertices$A(u|0)$, $B(t|0)$ and $C(u|f_t(u))$ has maximal area.

Exercise 11: Applications

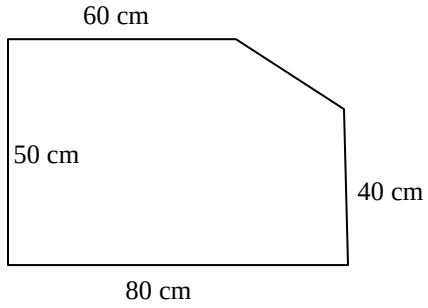
a) The cost of producing a coffee machine is 22 €. The marketing department predicts the sale of 60 000 machines at the price of 45 € Furthermore they guess that each price reduction by 1 € will result in 8000 additional machines sold. Find the price with maximal gain (= sales – cost)

b) A rectangular pen is partitioned into two also rectangular sections and built from 400 m of fence. What dimensions should be used so that the area will be a maximum?

c) A rectangular pen has an area of 400 m^2 and is bordered at one side by a straight canal. What dimensions should be used to minimize the length of the fence?

d) A rectangle with maximal area must be fitted into a triangle with base length 8 cm and height 6 cm. **Hint**: Show that it is sufficient to examine the special case of the **rectangular** triangle.

e) A triangular corner has broken off a rectangular window pane during transport. Find the rectangle with maximal area which can be cut out of the remaining form shown left.



f) A wooden beam with rectangular cross section of height y and width x has a carrying strength of $T = kxy^2$ with the material constant k. Find the dimensions of a rectangular beam made from a cylindrical trunk with radius r = 12 cm.

g) A cylinder with maximal volume is to be fitted into a ball with radius R = 5 cm. A cylinder with height h and radius r has the volume $V = \pi \cdot r^2 \cdot h$.

h) A factory produces tin cans with a content of 1 Litre. Find radius r and height h so that needed area of tin sheet becomes minimal.

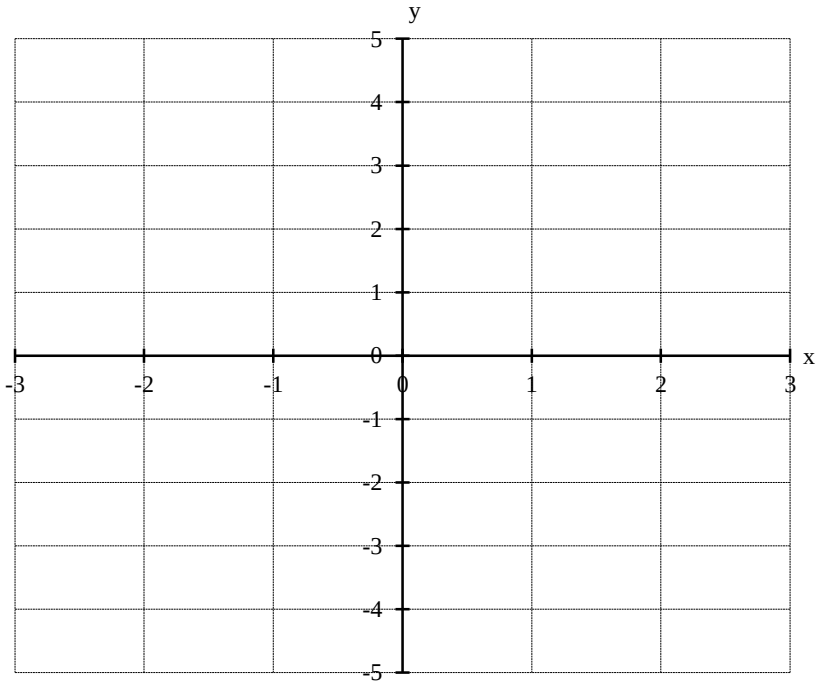
Exercise 12: Newton-Raphson-method

Find the first five estimates of the Newton-Raphson iteration for the given first guess x_0.

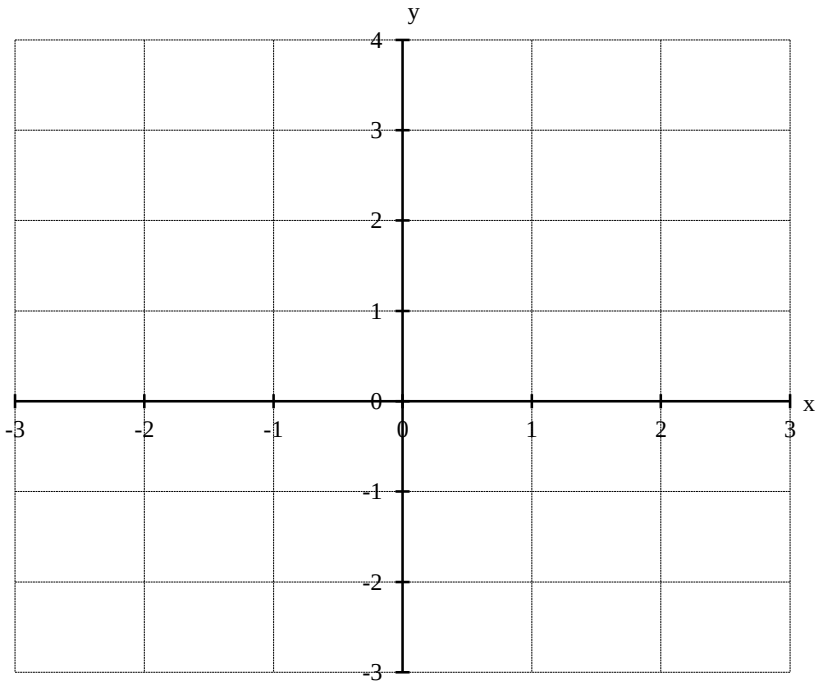
a) $f(x) = x^3 - x^2 + 3$ für $x_0 = 1$ b) $f(x) = 2x^4 - 4x^2 + 1$ für $x_0 = 2$ c) $f(x) = x^3 - x - 1$ für $x_0 = 1$

Coordinate systems for exercise 1

Exercise 1 a)

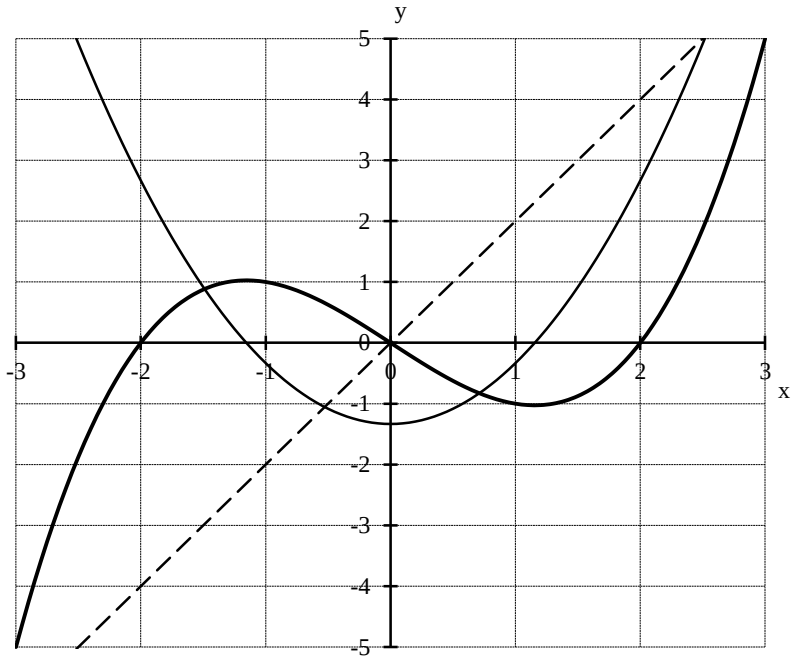


Exercise 1 b)

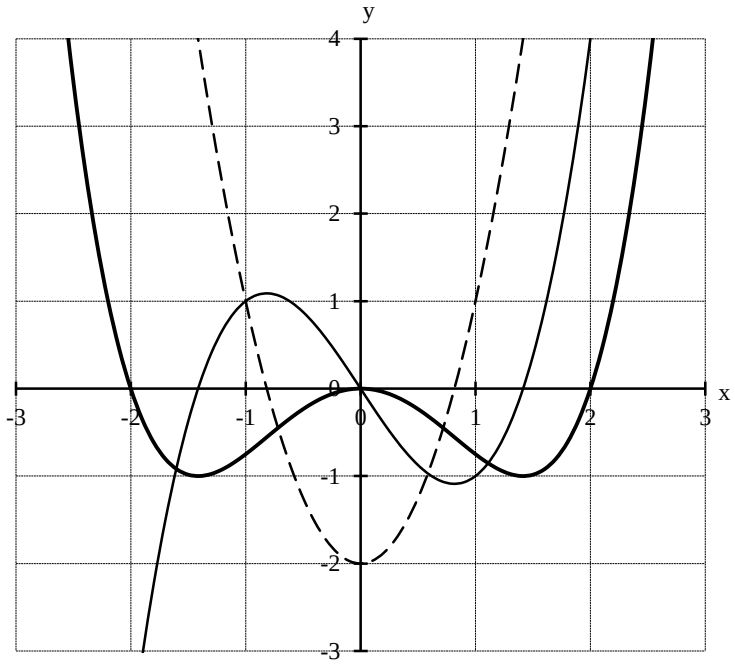


Graphs for exercise 1

Exercise 1 a)



Exercise 1 b)



5.3. Solutions to the exercises on curve analysis of polynomial functions

Exercise 1: Curve analysis

a) See script

b) $f(x) = \frac{1}{4}x^4 - x^2 = \frac{1}{4}\cdot x^2\cdot(x+2)\cdot(x-2)$

Symmetry: f is even with $f(-x) = f(x) \Rightarrow$ symmetry with respect to the y-axis

Intercepts $N_1(-2|0)$, $N_2(0|0)$ (double $\Rightarrow$ contact point), $N_3(2|0)$

Derivatives: $f'(x) = x^3 - 2x = x(x^2 - 2)$, $f''(x) = 3x^2 - 2$ and $f'''(x) = 6x$

Minima: $f(\pm\sqrt{2}) = -1$ (y-value), $f'(\pm\sqrt{2}) = 0$ (horizontal), $f''(\pm\sqrt{2}) = 4 > 0$ (convex) $\Rightarrow$ relative $Min_{1/2}(\pm\sqrt{2}|-1)$,

Maximum: $f(0) = 0$ (y-value), $f'(0) = 0$ (horizontal tangent line), $f''(0) = -2 < 0$ (concave) $\Rightarrow$ relative Max(0|0)

Inflexion points: $f(\pm\sqrt{\frac{2}{3}}) = -\frac{5}{9}$ (y-value), $f''(\pm\sqrt{\frac{2}{3}}) = 0$ with $f'''(\pm\sqrt{\frac{2}{3}}) \neq 0$ (sign change in f'') $\Rightarrow Inf_{1/2}(\pm\sqrt{\frac{2}{3}}|-\frac{5}{9})$

c) $f(x) = \frac{1}{3}(x^3 - 1)$

Symmetry: $f(\frac{1}{3} + x) = -f(\frac{1}{3} - x) \Rightarrow$ point symmetry with respect to $S_y(0|-\frac{1}{3})$

Intercepts: $S_x(1|0)$, $S_y(0|-\frac{1}{3})$

Derivatives: $f'(x) = x^2$, $f''(x) = 2x$ und $f'''(x) = 2$

Inflexion points: $f(0) = -\frac{1}{3}$, $f'(0) = 0$, $f''(0) = 0$, $f'''(0) = 2 > 0 \Rightarrow$ saddle point $S(0|-\frac{1}{3})$

d) $f(x) = \frac{1}{4}(x^4 - 1)$

Symmetry: f is even with $f(x) = f(-x) \Rightarrow$ Symmetry with respect to the y-axis

Intercepts: $S_{x1/2}(\pm1|0)$, $S_y(0|-\frac{1}{4})$

Derivatives: $f'(x) = x^3$, $f''(x) = 3x^2$ and $f'''(x) = 6x$

Extrema: $f(0) = -\frac{1}{4}$, (y-value), $f'(0) = 0$ (triple zero with sign change from − to +, i.e. transition from decreasing to increasing with horizontal tangent line), $f''(0) = 0$ (double zero without sign change, $f''(x) \neq 0$ for all $x \in \mathbb{R}$, i.e. convex on $\mathbb{R}$), $f'''(0) = 0 \Rightarrow$ relative (and absolute) minimum $T(0|-\frac{1}{4})$

e) $f(x) = \frac{1}{4}(x^5 - 1)$

Symmetry: $f(-\frac{1}{5} + x) = -f(-\frac{1}{5} - x) \Rightarrow$ Point symmetry with respect to $S_y(0|-\frac{1}{5})$

Intercepts: $S_x(1|0)$, $S_y(0|-\frac{1}{5})$

Derivatives: $f'(x) = x^4$, $f''(x) = 4x^3$ and $f'''(x) = 12x^2$

Extrema: $f(0) = -\frac{1}{5}$, (y-value), $f'(0) = 0$ (quadruple zero without sign change, $f'(x) \neq 0$, i.e. increasing on $\mathbb{R}$), $f''(0) = 0$ (triple zero with sign change from − to +, i.e. transition from concave to convex), $f'''(0) = 0 \Rightarrow$ inflexion point $W(0|-\frac{1}{5})$

f) $f(x) = -\frac{1}{3}x^3 + 3x = \frac{1}{3}x(x-3)(x+3)$

Symmetry: $f(-x) = -f(x) \Rightarrow$ symmetric to origin
Intercepts: S(0|0)
Derivatives: $f'(x) = -x^2 + 3$, $f''(x) = -2x$ and $f'''(x) = -2$
Extrema ($f'(x) = 0$ and $f''(x)$ <> 0): rel Max($-\sqrt{3}$ |$2\sqrt{3}$) and rel Min($\sqrt{3}$ |$-2\sqrt{3}$)
Inflexion points ($f''(x) = 0$ and $f'''(x) \neq 0$): Inf(0|0)

g) $f(x) = \frac{1}{6}x^3 - \frac{3}{2}x^2 + 4x = \frac{1}{6}x(x^2 - 9x + 24)$

Symmetry: none
Intercepts: S(0|0)
Derivatives: $f'(x) = \frac{1}{2}x^2 - 3x + 4 = \frac{1}{2}(x-2)(x-4)$, $f''(x) = x - 3$ and $f'''(x) = 1$
Extrema ($f'(x) = 0$ and $f''(x)$ <> 0): rel Max(2| $\frac{10}{3}$) and rel Min(4| $\frac{8}{3}$)
Inflexion points ($f''(x) = 0$ and $f'''(x) \neq 0$): Inf(3|3)

h) $f(x) = -\frac{1}{4}x^3 + x^2 = -\frac{1}{4}x^2(x-4)$

Symmetry: none
Intercepts: $S_{x1}(0|0)$ (double) and $S_{x2}(4|0)$
Derivatives: $f'(x) = -\frac{3}{4}x^2 + 2x = -\frac{3}{4}x(x - \frac{8}{3})$, $f''(x) = -\frac{3}{2}x + 2$ and $f'''(x) = -\frac{3}{2}$
Extrema ($f'(x) = 0$ and $f''(x)$ <> 0): rel Max($\frac{8}{3}$ | $\frac{64}{27}$) $\approx$ Max($2,\bar{6}$ |2,37) and rel Min(0|0)
Inflexion points ($f''(x) = 0$ and $f'''(x) \neq 0$): Inf($\frac{4}{3}$ | $\frac{32}{27}$) $\approx$ Inf($1,\bar{3}$ |1,16)

i) $f(x) = \frac{1}{12}x^4 - x^2 + 6 = \frac{1}{12}(x^4 - 12x^2 + 72)$

Symmetry: $f(-x) = f(x) \Rightarrow$ Symmetry with respect to the y-axis
Intercepts: S(0|6)
Derivatives: $f'(x) = \frac{1}{3}x^3 - 2x = \frac{1}{3}x(x^2 - 6)$, $f''(x) = x^2 - 2$ and $f'''(x) = 2x$
Extrema: $f'(x) = 0$ and $f''(x)$ <> 0 $\Rightarrow$ relative Max(0|6) and relative $Min_{1/2}$($\pm\sqrt{6}$ |3)
Inflexion points: $f''(x) = 0$ and $f'''(x) \neq 0 \Rightarrow Inf_{1/2}$($\pm\sqrt{2}$ | $\frac{13}{3}$)

j) $f(x) = -x^4 + 5x^2 - 4 = -(x^2 - 1)(x^2 - 4)$

Symmetry: $f(-x) = -f(x) \Rightarrow$ Symmetry with respect to y-axis.
Intercepts: $S_y(0|-4)$, $S_{1/2}(\pm1|0)$, $S_{3/4}(\pm2|0)$
Derivatives: $f'(x) = 4x^3 - 10x = 4x(x^2 - \frac{5}{2})$, $f''(x) = 12x^2 - 10 = 12(x^2 - \frac{5}{6})$, $f'''(x) = 24x$
Extrema: $f'(x) = 0$ and $f''(x)$ <> 0 $\Rightarrow$ Min(0|−4) and $Max_{1/2}$($\pm\sqrt{\frac{5}{2}}$ | $\frac{9}{4}$) $\approx$ $Max_{1/2}$(±1,58|2,25)
Inflexion points: $f''(x) = 0$ and $f'''(x) \neq 0 \Rightarrow Inf_{1/2}$($\pm\sqrt{\frac{5}{6}}$ |$-\frac{19}{36}$) $\approx$ $Inf_{1/2}$(±0,91|−0,53)

k) $f(x) = -\frac{1}{4}x^4 + 2x^2 + \frac{9}{4} = -\frac{1}{4}(x^2 - 9)(x^2 + 1)$

Symmetry: $f(-x) = f(x) \Rightarrow$ symmetry to y-axis

Intercepts: $S_y(0|\frac{9}{4})$ and $S_{x1/2}(\pm 3|0)$

Derivatives: $f'(x) = -x^3 + 4x = -x(x - 2)(x + 2)$, $f''(x) = -3x^2 + 4 = -3(x^2 - \frac{4}{3})$ and $f'''(x) = -6x$

Extrema ($f'(x) = 0$ and $f''(x)$ </> 0): rel $Max_{1/2}(\pm 2|\frac{25}{4}) = Max_{1/2}(\pm 2|6{,}25)$ and rel $Min(0|\frac{9}{4}) = Min(0|2{,}25)$

Inflexion points ($f''(x) = 0$ and $f'''(x) \neq 0$): $Inf_{1/2}(\pm\sqrt{\frac{4}{3}}\,|\,\frac{161}{36}) \approx Inf_{1/2}(\pm 1{,}155|4{,}472)$

l) $f(x) = -\frac{1}{20}x^5 + \frac{5}{6}x^3 - \frac{9}{4}x = -\frac{1}{20}x(x^4 - \frac{50}{3}x^2 + 45)$

Symmetry: $f(-x) = -f(x) \Rightarrow$ point symmetry with respect to origin

Intercepts: $S(0|0)$ and $S_{x1/2/3/4}(\pm\sqrt{\frac{1}{3}\ 25 \pm 10\sqrt{2{,}2}}\ |0) \approx S_{x1/2}(\pm 1{,}84|0)$ and $S_{x3/4}(\pm 3{,}64|0)$ (not asked)

Derivatives: $f'(x) = -\frac{1}{4}x^4 + \frac{5}{2}x^2 - \frac{9}{4} = -\frac{1}{4}(x^2 - 9)(x^2 - 1)$, $f''(x) = -x^3 + 5x$ and $f'''(x) = -3x^2 + 5$

Relative maxima ($f'(x) = 0$ and $f''(x) < 0$): $Max_1(-1|\frac{22}{15}) = Max_1(-1|1{,}4\overline{6})$ and $Max_2(3|\frac{18}{5}) = Max_2(3|3{,}6)$

Relative minima ($f'(x) = 0$ and $f''(x) > 0$): $Min_1(-3|-\frac{18}{5}) = Min_1(-3|-3{,}6)$ and $Min_2(1|-\frac{22}{15}) = Min_2(1|-1{,}4\overline{6})$

Inflexion points: $f''(x) = 0$ and $f'''(x) \neq 0 \Rightarrow Inf_1(0|0)$ and $Inf_{2/3}(\pm\sqrt{5}\,|\mp\frac{1}{2}\sqrt{5}) \approx Inf_{2/3}(\pm 2{,}24|\mp 1{,}12)$

m) $f(x) = \frac{1}{4}x^5 - \frac{25}{12}x^3 + 5x = \frac{1}{4}x(x^4 - \frac{25}{3}x^2 + 20)$

Symmetry: $f(-x) = -f(x) \Rightarrow$ point symmetry with respect to origin
Intercepts: $S_2(0|0)$

Derivatives: $f'(x) = \frac{5}{4}(x^4 - 5x^2 + 4) = \frac{5}{4}(x^2 - 1)(x^2 - 4)$, $f''(x) = 5x^3 - \frac{25}{2}x = 5x(x^2 - \frac{5}{2})$ und $f'''(x) = 15x^2 - 12$

Relative maxima ($f'(x) = 0$, $f''(x) < 0$): $Max_1(-2|-\frac{4}{3}) \approx Max_1(-2|-1{,}\overline{3})$ and $Max_2(1|\frac{38}{12}) = Max_2(1|3{,}1\overline{6})$,

Relative minima ($f'(x) = 0$, $f''(x) > 0$): $Min_1(-1|-\frac{38}{12}) = Min_1(-1|-3{,}1\overline{6})$ and $Min_2(2|\frac{4}{3}) \approx Min_2(2|1{,}\overline{3})$

Inflexion points ($f''(x) = 0$ and $f'''(x) \neq 0$): $Inf_1(0|0)$ and $Inf_{2/3}(\pm\sqrt{\frac{5}{2}}\,|\mp\frac{65}{48}\sqrt{\frac{5}{2}}) \approx Inf_{2/3}(\pm 1{,}58|\mp 2{,}14)$

n) $f(x) = \frac{1}{10}(x + 1)(x - 2)^3 = \frac{1}{10}(x^4 - 5x^3 + 6x^2 + 4x - 8)$

Symmetry: none
Intercepts: $S_y(0|-0{,}8)$, $S_{1x}(-1|0)$, $S_{2x}(2|0)$ (triple)

$f'(x) = \frac{1}{10}(4x^3 - 15x^2 + 12x + 4) = \frac{1}{10}(4x + 1)(x - 2)^2$, $f''(x) = \frac{3}{5}(2x^2 - 5x + 2) = \frac{3}{5}(2x + 1)(x - 2)$, $f'''(x) = \frac{3}{5}(4x - 5)$

Extrema: $f'(x) = 0$ und $f''(x) > 0 \Rightarrow Min(-\frac{1}{4}|-0{,}845)$

Saddle point: $f'(x) = 0$, $f''(x) = 0$ and $f'''(x) \neq 0 \Rightarrow S(2|0)$

Inflexion point: $f''(x) = 0$ and $f'''(x) \neq 0 \Rightarrow Inf(\frac{1}{2}|-0{,}506)$

Exercise 3: Common points of families of curves

a) $f_t(x) = x^2 + 2tx + 1 - 2t = (x + t)^2 - (t + 1)^2 + 2 \Rightarrow$ parabolas with vertices $S(-t|-(t + 1)^2 + 2)$ and $S_g(1|2)$

b) $f_t(x) = \frac{1}{x-t} + \frac{t}{t-1} \Rightarrow$ hyperbolas with symmetry centre $S(t|\frac{t}{t-1})$ and $S_g(1|0)$

c) $f_t(x) = x^3 + 3x^2 + (3 - t^2)x + 1 + t^2 = (x - 1 + t)(x - 1)(x - 1 - t)$ with $S_g(1|0)$

Exercise 3: Envelopes of families of lines

a) $f(x) = x^2$ b) $f(x) = \frac{1}{x}$ c) $f(x) = \sqrt{x}$

Exercise 4: Loci of extrema and inflexion points

a) see script

b) relative min $T_a(-3|-9+a)$ with $x = -3$

c) relative min $T_a\left(-\frac{a}{2}\middle|-\frac{a^2}{4}+2\right)$ with $y = -x^2 + 2$

d) relative min $T_a\left(\frac{4}{a}\middle|-\frac{16}{a}-5\right)$ with $y = -4x - 5$ for $a \neq 0$. For $a = 0$ the line $f_0(x) = -8x - 5$ results.

e) relative max $H_t\left(\frac{4}{t}\middle|t\right)$ with $y = \frac{4}{x}$ for $t \neq 0$. For $t = 0$ the horizontal line $f_0(x) = 0$ results

f) relative min $T_k(k|-\frac{1}{2}k^3 - 3k^2 + 2)$ with $y = -\frac{1}{2}x^3 - 3x^2 + 2$ and $H_k(-2|6k + 6)$ with $x = -2$ and

inflexion points $W(1-\frac{1}{2}k|4(1-\frac{1}{2}k)^3 - 6k(1-\frac{1}{2}k) + 2$ with $y = 4x^3 - 6kx + 2$

g) relative max $H_a\left(-\sqrt{a}\middle|\frac{2}{\sqrt{a}}\right)$ with $y = -\frac{2}{x}$ for $x > 0$ and relative min $T_a\left(\sqrt{a}\middle|-\frac{2}{\sqrt{a}}\right)$ with $y = -\frac{2}{x}$ for $x < 0$ and $a > 0$. For $a < 0$ there are no extrema. $W(0|0)$ remains stationary

h) relative max $H_t\left(t\middle|\frac{2}{3}t^2\right)$ with $y = \frac{2}{3}x^2$ and $T_t(3t|0)$ with $y = 0$ and inflexion points $W_t(2t|\frac{1}{3}t^2)$ with $y = \frac{1}{12}x^2$.

i) relative max $H(0|0)$ and rel. min $T_a\left(\frac{2}{3a}\middle|-\frac{4}{27a^2}\right)$ with $y = -\frac{1}{3}x^2$ and inflexion points $W_a\left(\frac{1}{3a}\middle|-\frac{2}{27a^2}\right)$ with $y = -\frac{2}{3}x^2$

j) relative min $T_b\left(-\frac{3}{4}b\middle|-\frac{27}{256}b^4\right)$ with $y = -\frac{1}{3}x^4$ and inflexion points $W_a\left(-\frac{1}{2}b\middle|-\frac{1}{16}b^4\right)$ with $y = x^4$.

k) relative min $T_t(-\sqrt{\frac{2}{3}}\frac{1}{t}|-\frac{4}{3}\sqrt{\frac{2}{3}}\frac{1}{t} + 1)$ with $y = \frac{4}{3}x + 1$ and rel. max $H_t(\sqrt{\frac{2}{3}}\frac{1}{t}|\frac{4}{3}\sqrt{\frac{2}{3}}\frac{1}{t} + 1)$ with $y = \frac{4}{3}x + 1$ and $W(0|1)$

l) Extrema $E_t(2t|\frac{1}{4t})$ with $y = \frac{1}{2x}$, rel max for $t > 0$ resp. rel min for $t < 0$ and inflexion points $W_t(3t|\frac{2}{9t})$ with $y = \frac{2}{3x}$

Exercise 5: Finding functions with given properties

a) $f(x) = ax^3 + bx^2 + cx + d$ with: Symmetry to O(0|0) ⇒ b = 0 and d = 0 $f(2) = 0 \Rightarrow 8a + 4b + 2c + d = 0$ $f'(0) = 2 \Rightarrow c = 2$ $\Rightarrow f(x) = -\frac{1}{2}x^3 + 2x$	b) $f(x) = ax^3 + bx^2 + cx + d$ with: $f(-2) = -2 \Rightarrow -8a + 4b - 2c + d = -2$ $f'(-2) = 0 \Rightarrow 12a - 4b + c = 0$ $f(0) = -4 \Rightarrow d = -4$ $f''(0) = 0 \Rightarrow b = 0$ $\Rightarrow f(x) = \frac{1}{8}x^3 - \frac{3}{2}x - 4$
c) $f(x) = ax^3 + bx^2 + cx + d$ with: $f'(0) = 0 \Rightarrow c = 0$ $f(3) = 3 \Rightarrow 27a + 9b + 3c + d = 0$ $f'(3) = 1 \Rightarrow 27a + 6b + c = 1$ $f''(3) = 0 \Rightarrow 18a + 2b = 0$ $\Rightarrow f(x) = -\frac{1}{27}x^3 + \frac{1}{3}x^2 + 1$	d) $f(x) = ax^3 + bx^2 + cx + d$ with $f(4) = 0: \Rightarrow 64a + 16b + 4b + d = 0$ $f(2) = 7: \Rightarrow 8a + 4b + 2c + d = 7$ $f'(2) = 0 \Rightarrow 12a + 4b + c = 0$ $f(1) = \frac{27}{4} \Rightarrow a + b + c + d = \frac{27}{4}$ $f'(1) = 0 \Rightarrow 3a + 2b + c = 0$ $\Rightarrow f(x) = -\frac{1}{2}x^3 + \frac{9}{4}x^2 - 3x + 8$
e) $f(x) = ax^3 + bx^2 + cx + d$ with: $f(1) = \frac{5}{4} \Rightarrow a + b + c + d = \frac{5}{4}$ $f''(1) = 0 \Rightarrow 6a + 2b = 0$ $f(-1) = g(-1) = \frac{13}{4} \Rightarrow -a + b - c + d = \frac{13}{4}$ $f'(-1) = 1 \Rightarrow 3a - 2b + c = 1$ $\Rightarrow f(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 - \frac{5}{4}x + 3$	f) $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with: Symmetry to y-axis ⇒ b = 0 and d = 0 $f(1) = 2 \Rightarrow a + b + c + d + e = 2$ $f'(1) = 0 \Rightarrow 4a + 3b + 2c + d = 0$ $f''(1) = 0 \Rightarrow 12a + 6b + 2c = 0$ $\Rightarrow f(x) = -\frac{1}{4}x^4 + \frac{3}{2}x^2 + \frac{3}{4}$
g) $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $f(2) = 0 \Rightarrow 16a + 8b + 4c + 2d + e = 0$ $f'(2) = 0 \Rightarrow 32a + 12b + 4c + d = 0$ $f''(2) = 0 \Rightarrow 48a + 12b + 2c = 0$ $f(0) = 0 \Rightarrow e = 0$ $f'(0) = 8 \Rightarrow d = 8$ $\Rightarrow f(x) = -x^4 + 6x^3 - 12x^2 + 8x$	h) $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ with: point symmetry to O(0|0) ⇒ b = d = f = 0, $f(2) = \frac{64}{25} \Rightarrow 32a + 16b + 8c + 4d + 2e + f = \frac{64}{25}$ $f'(2) = 0 \Rightarrow 80a + 32b + 12c + 4d + e = 0$ $f''(2) = 0 \Rightarrow 160a + 48b + 12c + 2d = 0$ $\Rightarrow f(x) = \frac{3}{100}x^5 - \frac{2}{5}x^3 + \frac{12}{5}x$
i) $f(x) = ax^3 + bx^2 + cx + d$ with $f(0) = 0 \Rightarrow d = 0$ and $f'(0) = \frac{3}{2}t \Rightarrow c = \frac{3}{2}t$ $f(3t) = 0: \Rightarrow 27t^3a + 9t^2b + 3tc + d = 0$ $f'(3t) = 0 \Rightarrow 27t^2a + 6tb + \frac{3}{2}t = 0$ $\Rightarrow f_t(x) = \frac{1}{6t}x^3 - x^2 + \frac{3}{2}tx$	j) $f(x) = ax^3 + bx^2 + cx + d$ with point symmetry to O(0|0) ⇒ b = d = 0 and $f(2) = 4 \Rightarrow 8a + 4b + 2c + d = 4$ $\Rightarrow f_a(x) = ax^3 + (2 - 4a)x$

Exercise 6: Optimization problems

a) <u>Area</u> $A(u) = \frac{1}{2}(f(u) - 0)(5 - u) = \frac{1}{8}(u^3 - 7u^2 - 5u + 75)$, $A'(u) = \frac{1}{8}(3u^2 - 14u - 5)$, $A''(u) = \frac{1}{4}(3u - 7) \Rightarrow$ rel max at $u = -\frac{1}{3}$ endpoints $A(0) = \frac{75}{8}$ and $A(1) = \frac{64}{8} \Rightarrow$ abs max at u = 0.

b) parabolas with vertices $S_f(2|-2)$ and $S_g(4|2)$ intersecting in $S_g(4|2)$ (right hand endpoint)

<u>Area</u> $A(u) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot(u-1)\cdot(f(u)-g(u)) = -2u^2 + 10u - 8$, $A'(u) = -4u + 10$, $A''(u) = -4 \Rightarrow$ relative max at $u = \frac{5}{2}$ with $A(\frac{5}{2}) = \frac{9}{2}$ endpoints: $A(1) = 0$ and $A(4) = 0 \Rightarrow$ absolute max at $u = \frac{5}{2}$.

Exercise 7: Optimization problems regarding distance and space between two graphs

a) <u>Area</u> $PQ(u) = g(u) - f(u) = -2u^2 + 6u$, $PQ'(u) = -4u + 6$, $PQ''(u) = -4 \Rightarrow$ relative max at $u = \frac{3}{2}$ (outside) with $PQ(\frac{3}{2}) = 4{,}5$ and endpoints $PQ(0) = 0$ and $PQ(1) = 4 \Rightarrow$ absolute max at $u = 1$.

<u>Fläche</u> $A(u) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot(u-0)\cdot(g(u)-f(u)) = -u^3 + 3u^2$, $A'(u) = -3u^2 + 6u$, $A''(u) = -6u + 6 \Rightarrow$ relative max at $u = 2$ with $A(2) = 4$ (outside) and endpoints $A(0) = 0$ and $A(1) = 2 \Rightarrow$ abs max at $u = 1$.

b) $g(x) = x(x-2)^2$ has rel max $H(\frac{2}{3} \mid \frac{32}{27})$ and rel min $T(0|2)$. Intersection points at $x_1 = 0$, $x_2 = 1$ und $x_3 = 4 \Rightarrow$ 2 domains:

Domain $0 \le u \le 1$:

<u>Area</u> $PQ(u) = g(u) - f(u) = u^3 - 5u^2 + 4u$, $PQ'(u) = 3u^2 - 10u + 4$, $PQ''(u) = 6u - 10 \Rightarrow$ rel max at $u = \frac{5}{3} - \sqrt{\frac{13}{9}} \approx 0{,}46$ with $PQ(0{,}46) \approx 0{,}88$ and endpoints: $PQ(0) = 0$ and $PQ(1) = 0 \Rightarrow$ abs max at $u \approx 0{,}46$.

<u>Area</u> $A(u) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot(u-0)\cdot(g(u)-f(u)) = \frac{1}{2}u^4 - \frac{5}{2}u^3 + 2u^2$, $A'(u) = 2u^3 - \frac{15}{2}u^2 + 4u$, $A''(u) = 6u^2 - 15u + 4 \Rightarrow$ relative max at $u = \frac{15}{8} - \sqrt{\frac{97}{64}} \approx 0{,}64$ with $A(0{,}64) \approx 0{,}25$ and endpoints $A(0) = 0$ and $A(1) = 0 \Rightarrow$ abs max at $u \approx 0{,}64$.

Domain $1 \le u \le 4$:

<u>Area</u> $PQ(u) = f(u) - g(u) = -u^3 + 5u^2 - 4u$, $PQ'(u) = -3u^2 + 10u - 4$, $PQ''(u) = -6u + 10 \Rightarrow$ rel max at $u = \frac{5}{3} + \sqrt{\frac{13}{9}} \approx 2{,}87$ with $PQ(2{,}87) \approx 6{,}06$ and endpoints $PQ(1) = 0$ and $PQ(4) = 0 \Rightarrow$ abs max at $u \approx 2{,}87$.

<u>Area</u> $A(u) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot(u-0)\cdot(f(u)-g(u)) = -\frac{1}{2}u^4 + \frac{5}{2}u^3 - 2u^2$, $A'(u) = -2u^3 + \frac{15}{2}u^2 - 4u$, $A''(u) = -6u^2 + 15u - 4 \Rightarrow$ rel max at $u = \frac{15}{8} + \sqrt{\frac{97}{64}} \approx 3{,}11$ with $A(3{,}11) \approx 9{,}08$ and endpoints $A(1) = 0$ and $A(4) = 0 \Rightarrow$ abs max at $u \approx 3{,}11$.

Exercise 8: Optimization problems without given coordinates

a) Rectangle must be symmetrical with respect to y-Axis with width 2u and height f(u) $\Rightarrow$ <u>Area</u> $A(u) = g\cdot h = (u-(-u))\cdot(f(u) - 0) = -2u^3 + 8u$, $A'(u) = -6u^2 + 8$, $A''(u) = -12u \Rightarrow$ relative max at $u = \sqrt{\frac{4}{3}} \approx 1{,}15$ with $A(1{,}15) \approx 6{,}16$ and endpoints $A(-2) = 0$ and $A(2) = 0 \Rightarrow$ abs max at $u \approx 1{,}15$. (

b) Rectangle must be symmetrical with respect to y-Axis with width 2u and height f(u) $\Rightarrow$ <u>Area</u> $A(u) = 2u\cdot(-f(u)) = -2u^5 + 10u^3 + 40u$, $A'(u) = -10u^4 + 30u^2 + 40$, $A''(u) = -40u^3 + 60u \Rightarrow$ rel max at $u = 2$ and endpoints: $2{,}23 \approx \sqrt{5} \le u \le \sqrt{\frac{5}{2} + \sqrt{\frac{105}{4}}} \approx 2{,}76$ with $A(\sqrt{5}) = 40\sqrt{5} \approx 89{,}44$ (width $2\sqrt{5}$ und height = 20) and $A(2{,}76) = 0 \Rightarrow$ abs max at $u = \sqrt{5}$.

c) <u>Area</u> $A(u) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot(u-(-u))\cdot(1 - f(u)) = -u^3 + 4u$, $A'(u) = -3u^2 + 4$, $A''(u) = -6u \Rightarrow$ relative max at $u = \sqrt{\frac{4}{3}} \approx 1{,}15$ with $A(1{,}15) \approx 3{,}44$ with endpoints: $A(0) = 0$ und $A(2) = 0 \Rightarrow$ absolute max at $u = \sqrt{\frac{4}{3}}$.

d) Intersection points ($f(x) = g(x)$): $S_{1/2}(\pm 3|-3)$ Rectangle must be symmetrical with respect to y-Axis with width 2u and height f(u) – g(u) f(u) $\Rightarrow$ <u>Area</u>: $A(u) = b(u)\cdot h(u) = 2u\cdot[f(u) - g(u)] = 2u\cdot[9 - u^2] = -2u^3 + 18u \Rightarrow$ relative max at $u = \sqrt{3}$ with $A(\sqrt{3}) = 12\cdot\sqrt{3}$. Endpoints $A(0) = 0$ and $A(3) = 0 \Rightarrow$ absolute max at $u = \sqrt{3}$

Exercise 9: Optimization problems with shift

a) $f(x + 2) = -x^2 + 1$. Area: $A(u) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot 2u\cdot f(2 + u) = -u^3 + u$, $A'(u) = -3u^2 + 1$ and $A''(u) = -6u \Rightarrow$ relative max at $u = \frac{1}{\sqrt{3}}$ with $A(\frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}}$. End points: $A(1) = 0$ and $A(0) = 0 \Rightarrow$ abs max at $u = \frac{1}{\sqrt{3}}$

b) Area $A(u) = g\cdot h = (2+u - (2-u))\cdot(f(2+u) - 0) = -2u^3 + 8u$, $A'(u) = -6u^2 + 8$, $A''(u) = -12u \Rightarrow$ rel max at $u = \sqrt{\frac{4}{3}} \approx 1{,}15$ with $A(1{,}15) \approx 6{,}16$ and end points $A(0) = 0$ and $A(4) = 0 \Rightarrow$ abs max at $u \approx 1{,}15$.

Exercise 10: Optimization problems with families of functions

Area: $A_t(u) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot(t - u)\cdot(0 - f_t(u)) = -\frac{1}{2}\cdot(t - u)\cdot u^2\cdot(u - t) = \frac{1}{2}u^4 - tu^3 + \frac{1}{2}t^2u^2$ with $A_t'(u) = 2u^3 - 3tu^2 + t^2u$

$\Rightarrow$ rel max at $u = \frac{1}{2}t$ mit $A_t(\frac{1}{2}t) = \frac{t^4}{32}$ and endpoints: $A_t(0) = A_t(t) = 0 \Rightarrow$ abs max at $u = \frac{1}{2}t$

Exercise 11: Applications

a) Sales numbers $A(x) = 60\,000 - 8\,000\cdot(x - 45) = -8\,000\cdot x + 420\,000$ with x = price in € $\Rightarrow$ gain $G(x) = (x - 22)\cdot A(x) = -8\,000\cdot x^2 + 596\,000x - 9\,240\,000 = -8\,000(x^2 - 74{,}5x + 1155) \Rightarrow$ relative maximum at vertex with x = 37,25 € and A(37,25) = 122 000 machines sold and gain G(37,25) = 1 860 500 €

b) Length u and width w with $400 = 2u + 3w \Leftrightarrow u = 200 - 1{,}5w \Rightarrow$ Area $A(w) = u\cdot w = w\cdot(200 - 1{,}5w) = -1{,}5w(w - \frac{400}{3})$ is maximal for width $w = \frac{200}{3} = 66,\bar{6}$ m and length u = 100 m with area $A = 6666,\bar{6}\,m^2$.

c) Length u and width w with $400 = u\cdot w \Rightarrow$ fence length $L(u) = 2\cdot u + \frac{400}{u}$ with $L'(u) = 2 - \frac{400}{u^2}$ is minimal for length $u = 10\sqrt{2}$ and width $w = 20\sqrt{2}$.

d) Assuming a right angled triangle with hypotenuse $y = 6 - \frac{3}{4}u \Rightarrow$ Length u and height $h = 6 - \frac{3}{4}u \Rightarrow$ Area $A(u) = u\cdot h = 6u - \frac{3}{4}u^2 = -\frac{3}{4}u(u - 8)$ is maximal for length u = 4 cm and height h = 3 cm with area A = 12 cm². Using cavalieri's principle (not needed in ist full extent) or an appropriate drawing one shows easily that shifting the vertex of the triangle parallel to the base line doesn't change the dimensions of the rectangle inside.

e) Area $A(u) = u\cdot(80 - 0{,}5u) = -0{,}5(u^2 - 160u)$ is maximal for length u = 80 cm and height 40 cm, i.e. the whole strip containing the missing triangle can be cut off!

f) Carrying strength: $T(x) = k\cdot x\cdot y^2 = k\cdot x\cdot(d^2 - x^2)$ (Pythagoras) $= k\cdot d^2\cdot x - k\cdot x^3$ with $T'(x) = kd^2 - 3kx^2 \Rightarrow$ rel max at $x = \frac{d}{\sqrt{3}}$ with $T(\frac{d}{\sqrt{3}}) = \frac{2kd^3}{3\sqrt{3}}$ and endpoints $T(0) = 0$ and $T(r) = 0 \Rightarrow$ abs max for $x = \frac{d}{\sqrt{3}}$

g) Volume: $V(h) = \pi\cdot r^2\cdot h = \pi\left(R^2 - \left(\frac{h}{2}\right)^2\right)\cdot h$ (Pythagoras) $= \pi\cdot R^2\cdot h - \frac{1}{4}\pi\cdot h^3 \Rightarrow$ rel max at $h = \frac{2}{\sqrt{3}}R$ with $V(\frac{2}{\sqrt{3}}R) = \frac{4\pi R^3}{3\sqrt{3}}$ and endpoints $V(0) = 0$ and $V(R) = 0. \Rightarrow$ abs. max for $h = \frac{2}{\sqrt{3}}R$.

h) Volume $V = \pi\cdot r^2\cdot h$, surface area $A(r) = 2\cdot\pi\cdot rh + 2\pi\cdot r^2 = \frac{2V}{r} + 2\pi\cdot r^2$ with $A'(r) = -\frac{2V}{r^2} + 4\pi r \Rightarrow$ rel max at $r = \sqrt[3]{\frac{V}{2\pi}}$ with $A\left(\sqrt[3]{\frac{V}{2\pi}}\right) = 3\sqrt[3]{2\pi V^2}$ endpoints $\lim_{r\to\infty} A(r) = \lim_{r\to 0} = 0 \Rightarrow$ abs min for $r = \sqrt[3]{\frac{V}{2\pi}} \approx 5{,}42$ cm and $h = \sqrt[3]{\frac{4V}{\pi}} = 2r \approx 10{,}84$ cm. Comparison: Radius of ball $r = \sqrt[3]{\frac{3V}{4\pi}} \approx 6{,}20$ cm

Exercise 12: Newton-Raphson-method

a) $x_0 = 1$, $x_1 = -2$, $x_3 = -1{,}4375$, $x_4 = -1{,}2130$, $x_5 = -1{,}1756$
b) $x_0 = 2$, $x_1 = 1{,}6458$, $x_3 = -1{,}4307$, $x_4 = -1{,}3312$, $x_5 = -1{,}3078$, $x_6 = 1{,}3066$
c) $x_0 = 1$, $x_1 = 1{,}5$, $x_3 = 1{,}3478$, $x_4 = 1{,}3253$, $x_5 = 1{,}3247$