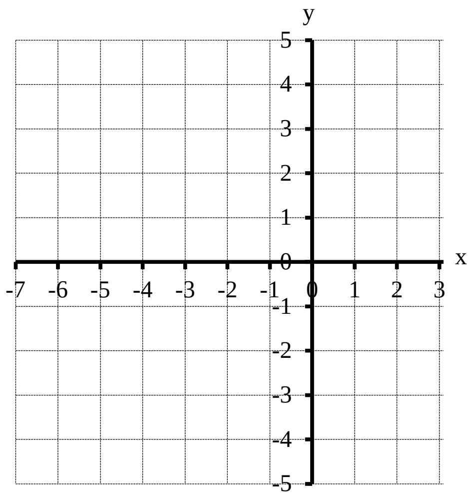
5.4. Exercises on the curve analysis of rational functions

Use Examples 22 b) – 25 b) with $f(x) = \dfrac{x^2-4}{x^2-1}$.

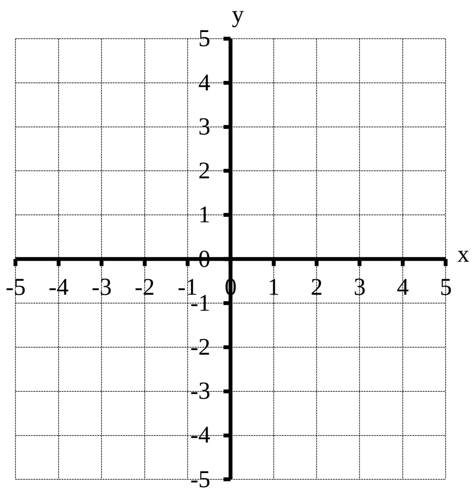
Exercise 1

a) Show that $f(x) = \dfrac{x-1}{x+2} = 1 - \dfrac{3}{x+2}$
b) Hence show that g(x) = 1 is a horizontal asymptote. Use limes notation
c) Find the vertical asymptote and the intercepts with x and y-axis
d) Find f'(x) and f''(x).
e) Show that there are no relative extrema and no inflexion points
f) Sketch the graph of f.



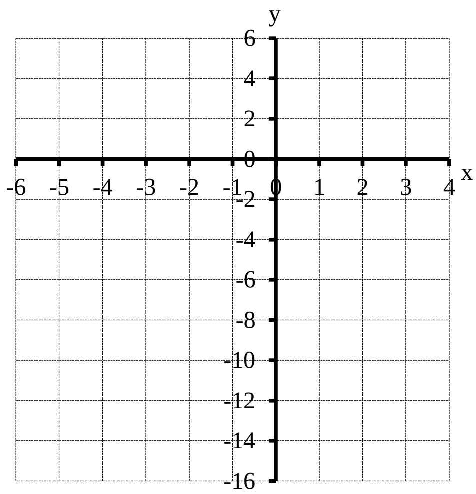
Exercise 2

a) Show that $f(x) = \dfrac{x^2-4x}{x^2-4} = 1 - \dfrac{4x-4}{x^2-4}$
b) Hence show that g(x) = 1 is a horizontal asymptote. Use limes notation
c) Find the vertical asymptotes and the intercepts with x and y-axis
d) Show that $f'(x) = \dfrac{4(x^2-2x+4)}{(x^2-4)^2}$
e) Hence show that f is increasing on its entire domain.
f) Sketch the graph of f including asymptotes.



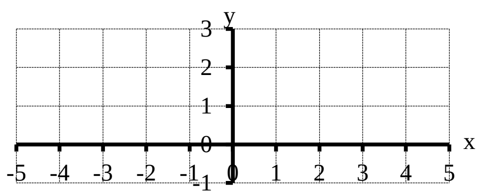
Exercise 3

a) Show that $f(x) = \dfrac{x^2-2x+1}{x+1} = x - 3 + \dfrac{4}{x+1}$
b) Hence show that g(x) = x − 3 is an asymptote. Use limes notation
c) Find the vertical asymptote and the intercepts with x and y-axis
d) Show that $f'(x) = \dfrac{(x+3)(x-1)}{(x+1)^2}$
e) Find f''(x)
f) Use the first derivative test to find the relative extrema
g) Sketch the graph including asymptotes.



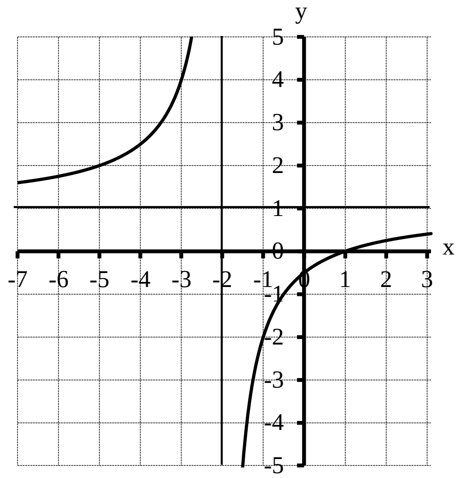
Exercise 4

a) Show that $f(x) = \dfrac{x^2-4x+4}{x^2+4} = 1 - \dfrac{4x}{x^2+4}$
b) Hence show that g(x) = 1 is a horizontal asymptote. Use limes notation
c) Find the the intercepts with x and y-axis
d) Show that $f'(x) = \dfrac{4(x^2-4)}{(x^2+4)^2}$
e) Show that $f''(x) = \dfrac{8x(x^2-12)}{(x^2+4)^3}$
f) Use the first derivative test to find the relative extrema
g) Use the second derivative test to find the inflexion points.



Solutions for the exercises on the curve analysis of rational functions

Exercise 1



a) $1 - \frac{3}{x+2} = \frac{x+2-3}{x+2} = \frac{x-1}{x+2} = f(x)$

b) $\lim_{x\to\pm\infty}\left(f(x)-g(x)\right) = \lim_{x\to\pm\infty}\frac{3}{x+2} = 0$

c) Vertical asymptote $x = -2$ (denominator vanishes);

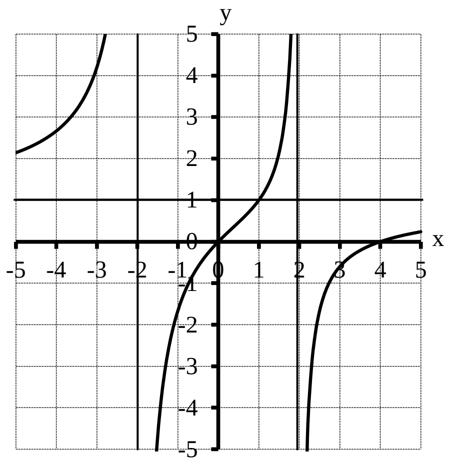
$S_y(0|-\frac{1}{2})$ (plug in x = 0) and $S_x(1|0)$ (numerator vanishes)

d) $f'(x) = \frac{3}{(x+2)^2}$ and $f''(x) = \frac{-6}{(x+2)^3}$

e) f' and f'' have no zeroes

f) Graph with asymptotes

Exercise 2



a) $1 - \frac{4x-4}{x^2-4} = \frac{x^2-4-(4x-4)}{x^2-4} = \frac{x^2-4x}{x^2-4} = f(x)$

b) $\lim_{x\to\pm\infty}\left(f(x)-g(x)\right) = \lim_{x\to\pm\infty}\frac{4x-4}{x^2-4} = 0$

c) Vertical asymptotes $x = \pm 2$ (denominator vanishes);
$S_y(0|0)$ (plug in x = 0) and $S_x(4|0)$ (numerator vanishes again)

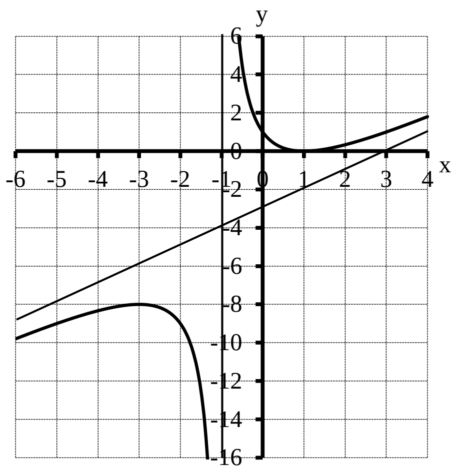
d) $f'(x) = -\frac{4(x^2-4)-(4x-4)\cdot 2x}{(x^2-4)^2} = -\frac{4x^2-16-8x^2+8x}{(x^2-4)^2}$

$= -\frac{-4x^2+8x-16}{(x^2-4)^2} = \frac{4(x^2-2x+4)}{(x^2-4)^2}$.

e) The numerator $4(x^2 - 2x + 4) = 4(x - 1)^2 + 12$ and the denominator $(x^2 - 4)^2$ of f' are both always greater than zero $\Rightarrow f'(x) > 0$ for all possible x.

f) Graph with asymptotes.

Exercise 3



a) $x - 3 + \frac{4}{x+1} = \frac{(x-3)(x+1)+4}{x+1} = \frac{x^2-2x-3+4}{x+1} = \frac{x^2-2x+1}{x+1} = f(x)$

b) $\lim_{x\to\pm\infty}\left(f(x)-g(x)\right) = \lim_{x\to\pm\infty}\frac{4}{x+1} = 0$

c) Vertical asymptotes $x = -1$ (denominator vanishes);
$S_y(0|1)$ (plug in x = 0) and $S_x(1|0)$ (double without sign change because of double zero in the numerator)

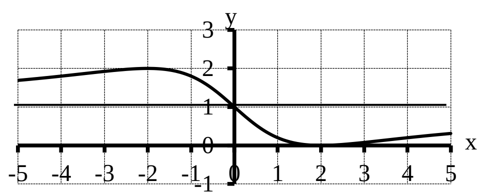
d) $f'(x) = 1 - \frac{4}{(x+1)^2} = \frac{(x+1)^2-4}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$

e) $f''(x) = \frac{8}{(x+1)^3}$

f) $f'(x) = 0 \Rightarrow x_1 = -3$ with $f''(-3) = -1 < 0$ and $f(-3) = -8 \Rightarrow \max(-3|-8)$
and $x_2 = 1$ with $f''(1) = 1 > 0$ and $f(1) = 0 \Rightarrow \min(1|0)$

g) Graph with asymptotes.

Exercise 4



a) $1 - \frac{4x}{x^2+4} = \frac{x^2+4-4x}{x^2+4} = \frac{x^2-4x+4}{x^2+4} = f(x)$

b) $\lim_{x\to\pm\infty}\left(f(x)-g(x)\right) = \lim_{x\to\pm\infty}\frac{4x}{x^2+4} = 0$

c) $S_y(0|1)$ (plug in x = 0) and $S_x(2|0)$ (double without sign change because of double zero in the numerator)

d) $f'(x) = -\frac{4(x^2+4)-4x\cdot 2x}{(x^2+4)^2} = \frac{-4x^2-16+8x^2}{(x^2+4)^2} = \frac{4(x^2-4)}{(x^2+4)^2}$