5.4. Exercises on the curve analysis of composite functions

Exercise 1: Curve analysis of exponential functions
Examine the function f with regard to intercepts, extrema, inflexion points, behaviour for $x \to \pm\infty$ and draw its graph.

a) $f(x) = x \cdot e^x$
b) $f(x) = x^2 \cdot e^{-x}$
c) $f(x) = e^{-x^2}$
d) $f(x) = x + e^{-\frac{x}{e}}$
e) $f(x) = -\frac{e}{2}x^2 + e^x$
f) $f(x) = \frac{1}{e}x - e^{x-2}$
g) $f(x) = \frac{2}{e}x + e^{-x^2}$
h) $f(x) = \frac{1}{2}(e^x + e^{-x})$
i) $f(x) = \frac{e^x}{e^x + e^{-x}}$

Exercise 2: Curve analysis of families of exponential functions
Examine the functions f_t for $t > 0$ with regard to intercepts, behaviour for $x \to \pm\infty$, extrema, inflexion points and draw their graphs for $t \in \{-2; 0; 2\}$.
Describe how the graph changes with increasing $t > 0$.
Describe the type of growth for $x \to \infty$: Is it linear, exponential or logistic growth?

a) $f_t(x) = \frac{x}{t} \cdot e^{-tx}$
b) $f_t(x) = e^{-(x-t)^2}$
c) $f_t(x) = \frac{e}{t}x - e^{\frac{x}{t}}$
d) $f_t(x) = e^{\frac{t \cdot x}{e}} - t \cdot x$
e) $f_t(x) = 100 - 100e^{-tx}$
f) $f_t(x) = \frac{100}{1 + 100 \cdot e^{-tx}}$

Exercise 3: Curve analysis of logarithm functions
Examine the function f with regard to intercepts, extrema, inflexion points, behaviour for $x \to \pm\infty$ and draw its graph.

a) $f(x) = \frac{\ln(x)}{x}$
b) $f(x) = (\ln(x))^2$
c) $f(x) = (\ln(x))^3$

Exercise 4: Curve analysis of rational functions
Examine the function f with regard to intercepts, extrema, inflexion points, behaviour for $x \to \pm\infty$ and draw its graph.

a) $f(x) = \frac{4x}{1 - x^2}$
b) $f(x) = \frac{(x+1)^2}{(x-1)^2}$
c) $f(x) = \frac{x^2 - 1}{x^2 + 1}$
d) $f(x) = \frac{x^2 - 7x + 11}{x - 5}$

Exercise 5: Curve analysis of families of rational functions
Examine the functions f_t for $t > 0$ with regard to intercepts, behaviour for $x \to \pm\infty$, extrema, inflexion points and draw their graphs for $t \in \{-2; 0; 2\}$.
Find the loci of all extrema and inflexion points.

a) $f_t(x) = x + \frac{4t^3}{(x-t)^2}$
b) $f_t(x) = x + t + \frac{1}{x-t}$
c) $f_t(x) = \frac{x-t}{x^2}$
d) $f_t(x) = \frac{x}{(x-t)^2}$
e) $f_t(x) = \frac{x+t}{(x-t)^2}$
f) $f_t(x) = x + 3t + \frac{4t^2}{x-t}$

Exercise 6: Mixed problems with families of functions

a) Find the loci of the maxima of $f_t(x) = tx^2 \cdot e^{-\frac{x}{t}}$ for t > 0. Give the formula of the symmetrical parabola connecting the two extrema.

b) Find the loci of all extrema and inflexion points of $f_t(x) = \frac{x}{t} \cdot e^{-tx}$ for t > 0.

c) Determine the intersection points and the intersection angles of the functions $f_t(x) = e^{tx}$ and $g_t(x) = e^{-\frac{x}{t}}$ for t > 0.

d) Calculate the area between the coordinate axes and the tangent line through the inflexion point of $f_t(x) = \frac{x}{t} \cdot e^{-tx}$ for t > 0.

e) Show that the area between the coordinate axes and the tangent line through the inflexion point of $f_t(x) = (t^2x + t) \cdot e^{-tx}$ for t > 0 does not depend on t.

f) For which values of t does the graph of $f_t(x) = x + e^{t-x}$ touch the x-axis ?

g) For which values of t exists an inflexion point of $f_t(x) = e^{tx} + k \cdot e^{(t+1)x}$? Calculate its coordinates..

h) Show that all graphs of $f_t(x) = e^{tx}$ für t > 0 meet at a common point P. The tangent and normal lines to the graphs of f_t in P together with the x-axis define a triangle. For which value of t will its area be minimal?

i) For which value of t does the intersection point of $f_t(x) = \frac{x + 2 + \ln(x+t)}{x+t}$ with the horizontal line y = 1 the smallest distance to the x-axis?

Exercise 7: Optimization problems with exponential functions

a) Which rectangle between the coordinate axes and the graph of $f(x) = e^{-x^2}$ has the maximal area?

b) Which rectangle between the positive coordinate axes and the graph of $f(x) = e^{-x}$ has the maximal area?

c) Which rectangle between the positive coordinate axes and the graph of $f(x) = 2x^2 \cdot e^x$ has the maximal area?

Exercise 8: Tangent and normal lines to exponential functions

a) The tangent line and the normal line to the graph of $f_t(x) = e^{tx}$ with t > 0 in P(0|1) together with the x-axis define a triangle. What is its largest possible area?

b) Examine the functions $f_t(x) = tx \cdot e^{-\frac{1}{2}(x^2-3)}$ with $t \in \mathbb{R}^+$ with regard to interceps, extrema and inflexion points. The tangent line to f_t in A($\sqrt{3}$ |t$\sqrt{3}$) together with the x-axis and the line $g_t(x) = \frac{1}{2t^2} \cdot x$ define a triangle. What is its largest possible area?

Exercise 9: Optimization problems with rational functions

a) Which rectangle with area A = 36 cm^2 has the smallest circumference??

b) Which sector with area A = 100 cm^2 has the smallest circumference?

c) Which point of the graph of $f(x) = \frac{2}{x^2}$ has the smallest distance to the origin? **Hint**: Because of the monotonicity of the root function $\sqrt{A}$ is minimal if and only if A is minimal. So it suffices to examine $x^2 + y^2$.

d) Which rectangle between the coordinate axes and the graph of $f(x) = \frac{20}{x^2+5}$ has the largest area?

e) The cross section of an underground sewer duct is a rectangle with a semicircle as roof and is to have an area of 8 m^2 Find its dimensions so that the number of bricks needed for the wall and roof is minimal.

f) Find the dimensions of a cylindrical 1 Litre tin can with minimal use of tin.

g) Find the dimensions of a 1 Litre tin can which consists of a cylindrical main body with hemispherical top an flat bottom so that the material use is minimal.

h) Find the dimensions of a cuboid tin container with quadratic cross section and minimal material use.

5.4. Solutions to the exercises on curve analysis of composite functions

Exercise 1: Curve analysis of exponential functions

a) $f(x) = x\cdot e^x$: No symmetry since $f(-x) \neq \pm f(x)$, intercept S(0|0), asymptote y = 0 since $f(x) \to 0$ for $x \to -\infty$. $f'(x) = (1 + x)\cdot e^x$, $f''(x) = (2 + x)\cdot e^x$ and $f'''(x) = (3 + x)\cdot e^x \Rightarrow$ Min$(-1|-\frac{1}{e}) \approx$ Min(−1|−0,368) and Inf$(-2|-\frac{2}{e^2}) \approx$ Inf(−2|−0,271)

b) $f(x) = x^2\cdot e^{-x}$: no symmetry since $f(-x) \neq \pm f(x)$, intercep: S(0|0) double $\Rightarrow$ contact point and min, since $f(x) \geq 0$ for all $x \in \mathbb{R}$; asymptote y = 0, since $f(x) \to$ for $x \to +\infty$. $f'(x) = (-x^2 + 2x)\cdot e^{-x}$, $f''(x) = (x^2 - 4x + 2)\cdot e^{-x}$ and $f'''(x) = (-x^2 + 6x - 6)\cdot e^x$ $\Rightarrow$ Min(0|0) and Max$(2|\frac{4}{e^2}) \approx$ Max(2|0,54) and $Inf_1(2-\sqrt{2}\,|f(2-\sqrt{2}\,)) \approx Inf_1(0{,}59|0{,}19)$ and $Inf_2(2+\sqrt{2}\,|f(2+\sqrt{2}\,)) \approx Inf_1(3{,}41|0{,}38)$

c) $f(x) = e^{-x^2}$: symmetry to y-axis since $f(-x) = f(x)$, intercept $S_y(0|1)$, asymptote y = 0, sinc $f(x) \to$ for $x \to +\infty$. $f'(x) = -2x\,e^{-x^2}$ and $f''(x) = (4x^2 - 2)\,e^{-x^2}$ $\Rightarrow$ Max(0|1) and $Inf_{1/2}(\pm\frac{1}{\sqrt{2}}\,|\,\frac{1}{\sqrt{e}})$

d) $f(x) = x + e^{-\frac{x}{e}}$: no symmetry since $f(-x) \neq \pm f(x)$, intercepts: $S_y(0|1)$ and $S_x(-e|0)$, asymptote g(x) = x for $x \to +\infty$, since $f(x) - g(x) = e^{-\frac{x}{e}} \to 0$ for $x \to +\infty$. $f'(x) = 1 - \frac{1}{e}\cdot e^{-\frac{x}{e}} = 1 - e^{-\frac{x}{e}-1}$ and $f''(x) = \frac{1}{e^2}\cdot e^{-\frac{x}{e}} = e^{-\frac{x}{e}-2}$ $\Rightarrow$ Min(−e|0)

e) $f(x) = -\frac{e}{2}x^2 + e^x$: no symmetry, since $f(-x) \neq \pm f(x)$, intercepts: $S_y(0|1)$ and $S_x(-0{,}627|0)$, asymptotic curve $g(x) = -\frac{e}{2}x^2$ for $x \to -\infty$ since $f(x) - g(x) = e^x \to 0$ for $x \to -\infty$. $f'(x) = -ex + e^x$, $f''(x) = -e + e^x$ and $f'''(x) = e^x \Rightarrow$ saddle point $S(1|\frac{e}{2}) \approx$ S(1|1,36)

f) $f(x) = \frac{1}{e}x - e^{x-2}$: no symmetry since $f(-x) \neq \pm f(x)$, intercepts: $S_y(0|-\frac{1}{e^2})$ and $S_x(1|0)$, asymptote: $g(x) = \frac{1}{e}x$ for $x \to -\infty$ since $f(x) - g(x) = e^{x-2} \to 0$ for $x \to -\infty$. $f'(x) = \frac{1}{e} - e^{x-2}$, $f''(x) = -e^{x-2}$ and $f'''(x) = -e^x - 2 \Rightarrow$ Max(1|0)

g) $f(x) = \frac{2}{e}x + e^{-x^2}$: no symmetry since $f(-x) \neq \pm f(x)$, intercepts: $S_y(0|1)$ and $S_x(-0{,}761|0)$, asymptote: $g(x) = \frac{2}{e}x$ for $x \to \pm\infty$ since $f(x) - g(x) = e^{-x^2} \to 0$ for $x \to \pm\infty$. $f'(x) = \frac{2}{e} - 2x\,e^{-x^2}$ and $f''(x) = (4x^2 - 2)\,e^{-x^2}$ $\Rightarrow$ Max$(1|\frac{3}{e})$ and $Inf_{1/2}(\pm\frac{1}{\sqrt{2}}\,|\pm\frac{1}{\sqrt{2}\cdot e} + \frac{1}{\sqrt{e}})$

h) $f(x) = \frac{1}{2}(e^x + e^{-x})$: symmetry to y-axis since $f(-x) = f(x)$) and $f(x) > 0$ for all $x \in \mathbb{R}$, intercepts: $S_y(0|1)$, no asymptotes. $f'(x) = \frac{1}{2}(e^x - e^{-x})$ and $f''(x) = \frac{1}{2}(e^x + e^{-x}) = f(x) \Rightarrow$ Min(0|1)

i) $f(x) = \frac{e^x}{e^x + e^{-x}}$: no symmetry since $f(-x) \neq \pm f(x)$, intercepts $S_y(0|\frac{1}{2})$, asymptotes: $g_1(x) = 0$ for $x \to -\infty$: and $g_2(x) = 1$ for $x \to +\infty$. $f'(x) = \frac{2}{(e^x + e^{-x})^2}$ and $f''(x) = \frac{-4(e^x - e^{-x})}{(e^x + e^{-x})^3}$ $\Rightarrow$ Inf$(0|\frac{1}{2})$

Exercise 2: Curve analysis of families of exponential functions

a) $f_t(x) = \frac{x}{t}\cdot e^{-tx}$: intercept S(0|0), asymptote y = 0 since $f(x) \to 0$ for $x \to +\infty$. $f_t'(x) = (\frac{1}{t} - x)e^{-tx}$ and $f_t''(x) = (tx - 2)\cdot e^{-tx}$ $\Rightarrow$ Max$(\frac{1}{t}\,|\,\frac{1}{et^2})$ and Inf$(\frac{2}{t}\,|\,\frac{1}{e^2t^2})$. Exponential shrinking with increasing rate for increasing t.

b) $f_t(x) = e^{-(x-t)^2}$: intercept $S(0|\,e^{-t^2})$, asymptote y = 0 since $f(x) \to 0$ for $x \to \pm\infty$. $f_t'(x) = -2(x - t)\,e^{-(x-t)^2}$ and $f_t''(x) = -2(1-2(x - t)^2)\,e^{-(x-t)^2}$ $\Rightarrow$ Max(t|1) and $Inf_{1/2}(t \pm \frac{1}{\sqrt{2}}\,|\,\frac{1}{\sqrt{e}})$. Exponential shrinking with increasing delay for increasing t

c) $f_t(x) = \frac{e}{t}x - e^{\frac{x}{t}}$: intercepts $S_x(t|0)$ und $S_y(0|-1)$, asymptote $g_t(x) = \frac{e}{t}x$ for $x \to -\infty$ since $f_t(x) - g_t(x) = e^{\frac{x}{t}} \to 0$ for $x \to -\infty$. $f_t'(x) = \frac{e}{t} - \frac{1}{t}e^{\frac{x}{t}}$ and $f_t''(x) = -\frac{1}{t^2}e^{\frac{x}{t}}$ ⇒ Max(t|0). Exponential growth with decreasing rate for increasing t.t.

d) $f_t(x) = e^{\frac{t\cdot x}{e}} - t\cdot x$: intercepts $S_x(\frac{e}{t}|0)$ and $S_y(0|1)$, asymptote $g_t(x) = tx$ for $x \to -\infty$ since $f_t(x) - g_t(x) = e^{\frac{tx}{e}} \to 0$ for $x \to -\infty$. $f_t'(x) = \frac{t}{e}e^{\frac{tx}{e}} - t$ and $f_t''(x) = \left(\frac{t}{e}\right)^2 e^{\frac{tx}{e}}$ ⇒ Max($\frac{e}{t}$|0). Exponential shrinking with increasing rate for increasing t.

e) $f_t(x) = 100 - 100e^{-tx}$: intercept S(0|0), asymptote g(x) = 100 for $x \to +\infty$ since $f_t(x) - g(x) = 100e^{-tx} \to 0$ for $x \to +\infty$. $f_t'(x) = 100te^{-tx}$ and $f_t''(x) = -100t^2e^{-tx}$. Limited growth with limit S = 100 with increasing rate for increasing t.

f) $f_t(x) = \frac{100}{1+100\cdot e^{-tx}}$: intercept $S_y(0|\frac{100}{101})$, asymptotes $g_1(x) = 0$ for $x \to -\infty$ since $f_t(x) \to 0$ for $x \to -\infty$ and $g_2(x) = 100$ for $x \to +\infty$ since $f_t(x) \to 100$ for $x \to +\infty$. $f_t'(x) = \frac{10000\cdot t\cdot e^{-tx}}{(1+100\cdot e^{-tx})^2}$ and $f_t''(x) = \frac{10000\cdot t^2\cdot e^{-tx}(100\cdot e^{-tx}-1)}{(1+100\cdot e^{-tx})^3}$ ⇒ Inf($\frac{\ln(100)}{t}$|50). Logistic growth with limit S = 100 with increasing rate for increasing t.

Exercise 3: Curve analysis of logarithm functions

a) $f(x) = \frac{\ln(x)}{x}$**:** intercept $S_x(1|0)$**,** asymptote**:** negative y-axis since $f(x) \to -\infty$ for $x \to 0^+$ and positive x-axis since $f(x) \to 0$ for $x \to +\infty$**.** $f'(x) = \frac{1}{x^2}(1 - \ln(x))$ and $f''(x) = \frac{1}{x^3}(2\ln(x) - 3)$ **⇒** Max(e|$\frac{1}{e}$) and Inf($e^{1,5}$|$\frac{1,5}{e^{1,5}}$) ≈ Inf(4,48|0,33)

b) $f(x) = (\ln(x))^2$: intercept: $S_x(1|0)$ (double ⇒ contact point), asymptotes: positive y-axis since $f(x) \to +\infty$ for $x \to 0^+$. $f'(x) = \frac{2\ln(x)}{x}$ and $f''(x) = \frac{2}{x^2}(1 - \ln(x))$ ⇒ Min(1|0) and Inf(e|1)

c) $f(x) = (\ln(x))^3$: intercept: $S_x(1|0)$ (triple), asymptote: negative y-axis since $f(x) \to -\infty$ for $x \to 0^+$. $f'(x) = \frac{3(\ln(x))^2}{x}$, $f''(x) = \frac{3}{x^2}\cdot\ln x\cdot(2 - \ln(x))$ ⇒ Inf(1|0) (saddle point) and Inf(e^2|8)

Exercise 4: Curve analysis of rational functions

a) $f(x) = \frac{4x}{1-x^2} = \frac{4x}{(x-1)(x+1)}$: Domain = $\mathbb{R}\backslash\{-1;1\}$, point symmetry to origin since f(−x) = −f(x), intercept: S(0|0), vertical asymptotes at x = ±1 since zeroes only in denominator, horizontal asymptote y = 0 for $x \to \pm\infty$ since degree of denominator > degree of numerator. $f'(x) = \frac{4(1+x^2)}{(1-x^2)^2}$ and $f''(x) = \frac{8x(3+x^2)}{(1-x^2)^3}$ ⇒ Inf(0|0)

b) $f(x) = \frac{(x+1)^2}{(x-1)^2} = 1 + \frac{4x}{(x-1)^2}$: Domain = $\mathbb{R}\backslash\{1\}$, no symmetry since f(−x) ≠ ± f(x), intercepts $S_y(0|1)$ and $S_x(-1|0)$, vertical asymptote without sign change at x = 1 since double zero only in the denominator, horizontal asymptote y = 1 for $x \to \pm\infty$ since $f(x) \to 1$ for $x \to \pm\infty$. $f'(x) = -\frac{4(x+1)}{(x-1)^3}$ and $f''(x) = \frac{8(x+2)}{(x-1)^4}$ ⇒ Min(−1|0) and Inf(−2|$\frac{1}{9}$)

c) $f(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$: Domain = $\mathbb{R}$, symmetry to y-axis since f(−x) = f(x), intercepts $S_y(0|-1)$ and $S_{x1/2}(\pm1|0)$, horizontal asymptote y = 1 for $x \to \pm\infty$ since $f(x) \to 1$ for $x \to \pm\infty$. $f'(x) = \frac{4x}{(x^2+1)^2}$ and $f''(x) = \frac{4-12x^2}{(x^2+1)^3}$ ⇒ Min(0|−1) and $Inf_{1/2}(\pm\frac{1}{\sqrt{3}}|-\frac{1}{2})$.

d) $f(x) = \frac{x^2-7x+11}{x-5} = x - 2 + \frac{1}{x-5}$, Domain = $\mathbb{R}\backslash\{5\}$, no symmetry, intercepts: $S_y(0|-\frac{11}{5})$ and $S_{x1/2}(\frac{7}{2} \pm \sqrt{\frac{5}{4}}|0)$, vertical asymptote at x = 5 since zero only in denominator, asymptotic line g(x) = x − 2 for $x \to \pm\infty$ since f(x) − g(x) = $\frac{2}{x^2+1} \to 0$ for $x \to \pm\infty$. $f'(x) = 1 - \frac{1}{(x-5)^2} = \frac{x^2-10x+24}{(x-5)^2}$ and $f''(x) = \frac{2}{(x-5)^3}$ ⇒ Max(4|1) and Min(6|5)

Exercise 5: Curve analysis of fmailies of rational functions

a) $f_t(x) = x + \frac{4t^3}{(x-t)^2} = \frac{x^3-2tx^2+t^2x+4t^3}{(x-t)^2}$ ⇒ Domain = $\mathbb{R}\backslash\{t\}$, $S_y(0|4t)$, $S_x(t|0)$, vertical asymptote without sign change at x = t and asymptotic line y = x. $f_t'(x) = 1 - \frac{8t^3}{(x-t)^3}$ and $f_t''(x) = \frac{24t^3}{(x-t)^4}$ ⇒ Max(3t|4t) with locus $y = \frac{4}{3}x$

b) $f_t(x) = x + t + \frac{1}{x-t} = \frac{x^2-t^2+1}{x-t}$ ⇒ Domain = $\mathbb{R}\backslash\{t\}$ with $S_y(0|-\frac{1}{t})$, $S_x(\pm\sqrt{t^2-1}|0)$ without sign change only for $|t| \geq 1$, vertical asymptote with sign change at x = t and asymptotic line y = x + t. $f_t'(x) = 1 - \frac{1}{(x-t)^2}$ and $f_t''(x) = \frac{2}{(x-t)^3}$ ⇒ Max(t − 1|2t − 2) with locus y = 2x and Min(t + 1|2t + 2) with the same locus y = 2x (!)

c) $f_t(x) = \frac{x-t}{x^2} = \frac{1}{x} - \frac{t}{x^2}$ ⇒ Domain = $\mathbb{R}\backslash\{0\}$ with $S_x(t|0)$ with sign change, vertical asymptote without sign change at x = 0 and horizontal asymptote y = 0. $f_t'(x) = -\frac{1}{x^2} + \frac{2t}{x^3} = \frac{2t-x}{x^3}$ and $f_t''(x) = \frac{2}{x^3} - \frac{6t}{x^4} = \frac{2x-6t}{x^4}$ ⇒ Max($2t|\frac{1}{4t}$) with locus $y = \frac{1}{2x}$ and Inf($3t|\frac{2}{9t^2}$) with locus $y = \frac{2}{x^2}$.

d) $f_t(x) = \frac{x}{(x-t)^2}$ ⇒ Domain = $\mathbb{R}\backslash\{t\}$ with $S_y(0|0)$ with sign change, vertical asymptote without sign change at x = t and horizontal asymptote y = 0. $f_t'(x) = -\frac{x+t}{(x-t)^3}$ and $f_t''(x) = \frac{2x+4t}{(x-t)^4}$ ⇒ Min($-t|-\frac{1}{4t}$) with locus $y = \frac{1}{4x}$ and Inf($-2t|-\frac{2}{9t}$) with locus $y = \frac{4}{9x}$.

e) $f_t(x) = \frac{x+t}{(x-t)^2}$ ⇒ Domain = $\mathbb{R}\backslash\{t\}$ with $S_y(0|\frac{1}{t})$, $S_x(-t|0)$ with sign change, vertical asymptote without sign change at x = t and horizontal asymptote y = 0. $f_t'(x) = -\frac{x+3t}{(x-t)^3}$ and $f_t''(x) = \frac{2x-8t}{(x-t)^4}$ ⇒ Min($-3t|-\frac{1}{8t}$) with locus $y = \frac{3}{8x}$ and Inf($4t|\frac{5}{4t}$) with locus $y = \frac{5}{x}$.

f) $f_t(x) = x + 3t + \frac{4t^2}{x-t} = \frac{(x+t)^2}{x-t}$ ⇒ Domain D = $\mathbb{R}\backslash\{t\}$ with $S_y(0|-1)$, $S_x(-t|0)$ with sign change, vertical asymptote with sign change at x = t and asymptotic line y = x + 3t. $f_t'(x) = 1 - \frac{4t^2}{(x-t)^2} = \frac{x^2-2tx-3t^2}{(x-t)^2}$ and $f_t''(x) = \frac{8t^2}{(x-t)^3}$ ⇒ Max(−t|0) with locus y = 0 and Min(3t|8t) with locus $y = \frac{8}{3}x$.

Exercise 6: Mixed problems with families of functions

a) $f_t'(x) = (2tx - x^2)\cdot e^{-\frac{x}{t}}$ and $f_t''(x) = (\frac{1}{t}x^2 - 4x + 2t)\cdot e^{-\frac{x}{t}}$ ⇒ Min(0|0) and $Max_t(2t|\frac{4t^3}{e^2})$ with locus $y = \frac{1}{2e^2}x^3$ and symmetrical parabola through T und H_t has formula $p_t(x) = \frac{t}{e^2}x^2$.

b) $f_t'(x) = (\frac{1}{t} - x)\cdot e^{-tx}$ and $f_t''(x) = (tx - 2)\cdot e^{-tx}$ ⇒ Max($\frac{1}{t}|\frac{1}{te}$) with locus $y = \frac{1}{e}x$ and Inf($\frac{2}{t}|\frac{2}{t^2e^2}$) with locus $y = \frac{2}{e^2}x^2$

c) Intersection point $S_{fg}(0|1)$ with angle $\alpha_{fg} = 90°$

d) $f_t'(x) = (\frac{1}{t} - x)\cdot e^{-tx}$ and $f_t''(x) = (tx - 2)\cdot e^{-tx} \Rightarrow Inf_t(\frac{2}{t} \mid \frac{2}{t^2e^2})$ with tangent line $w_t(x) = -\frac{1}{te^2}x + \frac{4}{t^2e^2}$ with intercepts $S_{yt}(0 \mid \frac{4}{t^2e^2})$ and $S_{xt}(\frac{4}{t} \mid 0) \Rightarrow$ area $A_t = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot\frac{4}{t^2e^2}\cdot\frac{4}{t} = \frac{8}{t^3e^2}$

e) $f_t'(x) = -t^3x\cdot e^{-tx}$ and $f_t''(x) = (t^4x - t^3)\cdot e^{-tx} \Rightarrow W(\frac{1}{t} \mid \frac{2t}{e})$with tangent line $w_t(x) = -\frac{t^2}{e}\cdot x + \frac{3t}{e}$ with $S_{tx}(\frac{3}{t} \mid 0)$ and $S_{yt}(0 \mid \frac{3t}{e})$ $\Rightarrow$ area $A_t = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot\frac{3}{t}\cdot\frac{3t}{e} = \frac{9}{2e}$.

f) $f_t'(x) = 1 - e^{t-x}$ and $f_t''(x) = e^{t-x} \Rightarrow Min(t \mid t + 1)$ touches x-axis for $t = -1$.

g) $f_t'(x) = te^{tx} + t(t + 1)\cdot e^{(t+1)x}$ and $f_t''(x) = t^2e^{tx} + t(t + 1)^2e^{(t+1)x} = [t + (t + 1)^2e^x]te^{tx} \Rightarrow$ the condition $f_t''(x) = 0$ is met if $x_t = \ln\left(-\frac{t}{(t+1)^2}\right)$ only for $t > 0$.

h) Common point $P(0 \mid 1)$. The tangent lines $t_t(x) = tx + 1$ and the normal lines $n_t(x) = -\frac{1}{t}x + 1$ through P intersect the x-axis in $S_{tt}(-\frac{1}{t} \mid 0)$ resp. $S_{nt}(t \mid 0)$. The triangle $S_{tt}PS_{nt}$ has the area $A(t) = \frac{1}{2}\cdot g\cdot h = \frac{1}{2}\cdot(t + \frac{1}{t})\cdot 1$ with $A'(t) = \frac{1}{2}(1 - \frac{1}{t^2})$ and $A''(t) = \frac{1}{t^3}$. A(t) attain rel. Min ($A'(t) = 0$ and $A''(t) > 0$) for $t = 1$. Since $\lim_{t\to\infty} A(t) = \lim_{t\to 0} A(t)$ it is a global Minimum.

i) Approach $f_t(x(t)) = 1 \Leftrightarrow x + 2 + \ln(x+2) = x + t \Leftrightarrow x(t) = e^{t-2} - t$ with $x'(t) = e^{t-2} - 1$ and $x''(t) = e^{t-2} \Rightarrow$Minimum of x at $t = 2$, since $x'(2) = 0$ and $x''(t) > 0$

Exercise 7: Optimization problems with exponential functions

a) $A(u) = b\cdot h = 2u\cdot f(u) = 2u\cdot e^{-u^2}$ with $u > 0$, $A'(u) = 2(1 - 2u^2)\cdot e^{-u^2}$ and $A''(u) = 4\cdot(2u^3 - 3u)\cdot e^{-u^2} \Rightarrow$ rel Max at $u = \frac{1}{2}\sqrt{2}$ with $A(\frac{1}{2}\sqrt{2}) = \sqrt{\frac{2}{e}}$ and $A(u) \to 0$ for $u \to 0$ and $u \to \infty \Rightarrow$ abs Max at $u = \frac{1}{2}\sqrt{2}$.

b) $A(u) = b\cdot h = u\cdot f(u) = u\cdot e^{-u}$ with $u > 0$, $A'(u) = (1 - u)\cdot e^{-u}$ and $A''(u) = (u - 2)\cdot e^{-u} \Rightarrow$ rel Max at $u = 1$ with $A(1) = e^{-1}$ and $A(0) = \lim_{u\to\infty} A(u) = 0 \Rightarrow$ abs Max at $u = 1$.

c) $A(u) = b\cdot h = u\cdot f(u) = (2u - u^3)\cdot e^u$ with $0 \le u \le -1 + \sqrt{3}$, $A'(u) = (2 + 2u - 3u^2 - u^3)\cdot e^u = (u - 1)\cdot(-u^2 - 4u - 2)\cdot e^u$ and $A''(u) = (2 + 4u - 9u^2 - 4u^3)\cdot e^u \Rightarrow$ rel Max at $u_1 = 1$ ($u_{2/3} = -2 \pm \sqrt{2} < 0$) with $A(1) = e$ and $A(0) = 0$ and $A(-1 + \sqrt{3}) \approx 2{,}23 < e \approx 2{,}71828 \Rightarrow$ abs Max at $u = 1$.

Exercise 8: Tangent and normal lines to exponential functions

a) Tangent line $t_t(x) = tx + 1$ with $S_{tx}(-\frac{1}{t} \mid 0)$ and normal line $n_t(x) = -\frac{1}{t}x + 1$ with $S_x(t \mid 0)$ define togenther with the x-axis a triangle with base $g(t) = t + \frac{1}{t}$ and height $h = 1$. Its area is $A(t) = \frac{1}{2}gh = \frac{1}{2}(t + \frac{1}{t})$ with $A'(t) = \frac{1}{2}(1 - \frac{1}{t^2})$. The area is minimal at $t = 1$ (assign change of first derivative from – to +) with $A(1) = 1$.

b) Point symmetry to origin, $f_t'(x) = -t(x^2 - 1)\cdot e^{-\frac{1}{2}(x^2-3)}$, $f_t''(x) = tx(x^2 - 3)\cdot e^{-\frac{1}{2}(x^2-3)} \Rightarrow Max(1 \mid te)$, $Min(-1 \mid -te)$, $Inf_{1/3}(\mp\sqrt{3} \mid \pm t\sqrt{3})$ and $Inf_2(0 \mid 0)$ (single zero with sign change in f_t''!) Triangle OPQ with $O(0 \mid 0)$, $P(\frac{3}{2}\sqrt{3} \mid 0)$ and $Q(\frac{6\sqrt{3}t^3}{1+4t^3} \mid \frac{3\sqrt{3}t}{1+4t^3})$ has area $A(t) = \frac{1}{2}\cdot b\cdot h = \frac{27}{4}\cdot\frac{t}{1+4t^3}$ with $A'(t) = \frac{27}{4}\cdot\frac{1-8t^3}{(1+4t^3)^2} \Rightarrow$ rel Max (zero in A' with sign change from + to −) at $t = \frac{1}{2}$ with $A(\frac{1}{2}) = \frac{9}{4}$, $A(0) = \lim_{t\to\infty} A(t) = 0 \Rightarrow$ abs Max at $t = \frac{1}{2}$.

Exercise 9: Optimization problems with rational functions

a) Circumference C = 2a + 2b with area A = a·b = 36 ⇒ $C(a) = 2a + \frac{72}{a}$ with $C'(a) = 2 - \frac{72}{a^2} = 0 \Leftrightarrow a = \pm 6$, but Domain $D_A =]0; \infty[$. Since C(a) → ∞ for a → 0 and a → ∞ the absolute Min is at a = b = 6 cm with C(6) = 24 cm.

b) Circumference C = 2r + s with area $A = \frac{1}{2} \cdot s \cdot r = 100$ ⇒ $C(r) = 2r + \frac{200}{r}$ with $C'(r) = 2 - \frac{200}{r^2} = 0 \Leftrightarrow r = \pm 10$, but Domain $D_A =]0; \infty[$. Since C(r) →∞ for r → 0 and r → ∞ the absolute Min is at r = 10 cm and s = 20 cm with C = 40 cm.

c) The area $A(x) = \sqrt{x^2 + \frac{4}{x^4}}$ is minimal if and only if $B(x) = x^2 + \frac{4}{x^4}$ is minimal. $B'(x) = 2x - \frac{16}{x^5} = 0$ yields $x = \pm\sqrt{2}$ with Domain $D_A =]0; \infty[$. Since B(x) → ∞ for x → 0 and x → ∞ the absolute Min is at $x = \sqrt{2}$ with $B(\sqrt{2}) = 3$ cm^2 resp. $A(\sqrt{2}) = \sqrt{3}$ cm.

d) Area $A(x) = 2x \cdot f(x) = \frac{40x}{x^2+5}$ with $A'(x) = \frac{200 - 40x^2}{(x^2+5)^2} = 0 \Rightarrow x = \pm\sqrt{5}$ with Domain $D_A =]0; \infty[$. Since A(x) → ∞ for x → 0 and x → ∞ the absolute Max is at $x = \sqrt{5}$ cm with $A(\sqrt{5}) = 4\sqrt{5}$ cm^2.

e) Circumference C = 2b + 2r + πr with Area $A = 2rb + \frac{\pi}{2}r^2 = 8 \Leftrightarrow b = \frac{4}{r} - \frac{\pi}{4}r \Rightarrow C(r) = \frac{\pi}{2}r + 2r + \frac{8}{r}$ with $C'(r) = \frac{\pi}{2} + 2 - \frac{8}{r^2} = 0$ for $r = \pm\frac{4}{\sqrt{4+\pi}} \approx \pm 1{,}5$. Since $A = 2rb + \frac{\pi}{2}r^2 = 8$ we have $r \le \frac{4}{\sqrt{\pi}}$, so the Domain is $D_C =]0; \frac{4}{\sqrt{\pi}}] \approx]0; 2{,}26]$. Since C(r) → ∞ for r → 0^+ and $C(\frac{4}{\sqrt{\pi}}) = 4\sqrt{\pi} + \frac{8}{\sqrt{\pi}} \approx 11{,}6$ m the absolute Min is at $r = \frac{4}{\sqrt{4+\pi}} \approx 1{,}5$ m with $C(\frac{4}{\sqrt{4+\pi}}) = 4\sqrt{4+\pi} \approx 10{,}69$ m.

f) Surface area S = 2πrh + 2πr^2 with Volume V = πr^{2}h = 1000 cm^3 ⇔ $h = \frac{1000}{\pi r^2} \Rightarrow S(r) = 2\pi r^2 + \frac{2000}{r}$ with $S'(r) = 4\pi r - \frac{2000}{r^2} = 0$ for $r = \sqrt[3]{\frac{500}{\pi}} \approx 5{,}42$ cm, with Domain $D_S =]0; \infty[$. Since S(r) → ∞ für r → 0 and r → ∞ the absolute Min is at $r = \sqrt[3]{\frac{500}{\pi}} \approx 5{,}42$ cm and h = 10,86 cm with $S(\sqrt[3]{\frac{500}{\pi}}) \approx 553{,}58$ cm^2. As comparison: A 1 Litre cube has a side length a = 10 cm and a surface area of $S_W = 6a^2 = 600$ cm^2!

g) Surface area $S = S_{Hemisphere} + S_{Bottom} + S_{lateral\ surface} = 2\pi r^2 + \pi r^2 + 2\pi rh = 3\pi r^2 + 2\pi rh$ with Volume $V = \frac{2}{3}\pi r^3 + \pi r^2 h = 1000$ cm^3 ⇒ $h = \frac{1000}{\pi r^2} - \frac{2r}{3} \Rightarrow S(r) = \frac{5}{3}\pi r^2 + \frac{2000}{r}$ with $S'(r) = \frac{10}{3}\pi r - \frac{2000}{r^2} = 0$ for $r = \sqrt[3]{\frac{600}{\pi}}$ cm ≈ 5,76 cm with Domain $D_S =]0; \sqrt[3]{\frac{1500}{\pi}}] =]0; 10\sqrt[3]{\frac{1{,}5}{\pi}}] \approx]0$ cm; 7,81 cm[. Since S(r) → ∞ for r → 0^+ and $S(10\sqrt[3]{\frac{1{,}5}{\pi}}) = 300\sqrt[3]{2{,}25\pi}$ cm^2 ≈ 575,7 cm^2 the absolute Min is at $r = \sqrt[3]{\frac{600}{\pi}}$ cm ≈ 5,76 cm and h ≈ 26,3 cm with $S(\sqrt[3]{\frac{600}{\pi}}) = 519{,}2$ cm^2 .

h) Surface area S = 2a^2 + 4ah with Volume V = a^{2}h = 1000 cm^3 ⇒ $h = \frac{1000}{a^2} \Rightarrow S(a) = 2a^2 + \frac{4000}{a}$ with $S'(a) = 4a - \frac{4000}{a^2} = 0$ for a = 10 cm, with Domain $D_S =]0; \infty[$. Since S(a) → ∞ for a → 0^+ and a → ∞, the absolute Min of S is at a = 10 cm and h = 10 cm with O(10) = 600 cm^2.