5.4. Exercises on differentiation rules

Exercise 1: Compound functions
Give the outer function g(x) and the inner function z(x)for the composition (x) = g(z(x)).

a) $f(x) = (2 + x)^5$ b) $f(x) = 1 - \sqrt{x}$ c) $f(x) = 2^{2x-1}$ d) $f(x) = \frac{1}{x^2+1}$

Exercise 2: Chain rule
Find the derivative:

a) $f(x) = (x^2 + 1)^3$ d) $f(x) = \sqrt{2x^2 + x - 3}$ g) $f(x) = \frac{1}{-x^3 + 6x - 4}$

b) $f(x) = (2x^2 + 3x - 1)^3$ e) $f(x) = \frac{1}{x-1}$ h) $f(x) = \sin(x^2 - 3x)$

c) $f(x) = \sqrt{3x-1}$ f) $f(x) = \frac{1}{x^2-1}$ i) $f(x) = \cos(x^3 + 1)$

Exercise 3: Chain rule with exponential functions
Find the derivative:

a) $f(x) = 2^x$ b) $f(x) = 10^{-x}$ c) $f(x) = e^{x^2-1}$ d) $f(x) = e^{3x-1}$ e) $f(x) = e^{-x^2}$

Exercise 4: Product rule
Find the derivative:

a) $f(x) = x\cdot e^x$ c) $f(x) = x^2\cdot\sin(x)$ e) $f(x) = x^3\cdot\cos(x)$
b) $f(x) = (3x^2 + x - 2)\cdot e^x$ d) $f(x) = (x^2 - 2)\cdot\sin(x)$ f) $f(x) = (x^3 - 2x + 1)\cdot\cos(x)$

Exercise 5: Chain rule and product rule
Find the derivative:

a) $f(x) = x\cdot e^{-x}$ c) $f(x) = (x^2 - 2)\cdot e^{2x-1}$ e) $f(x) = (2x^2 + 3)\cdot 3^x$ g) $f(x) = (x^2 - 2)\cdot\sqrt{3x-1}$

b) $f(x) = (x^2 - 2)\cdot e^{-x}$ d) $f(x) = x\cdot e^{x^2-1}$ f) $f(x) = x\cdot 2^{-x}$ h) $f(x) = \frac{x}{x-1}$

Exercise 6: Chain rule and product rule with logarithm functions
Find the derivative:
a) $f(x) = \ln(2x - 3)$ c) $f(x) = (2x - 1)\cdot\ln(x + 1)$ e) $f(x) = x\cdot\ln(x) - x$
b) $f(x) = x\cdot\ln(x^2 + 3)$ d) $f(x) = (2x - 2)\cdot\ln(x)$ f) $f(x) = \log_2(3x + 1)$

Exercise 7: Quotient rule
Find the derivative:

a) $f(x) = \frac{x^2+1}{2x+3}$ d) $f(x) = \frac{x^2+1}{2x+1}$ g) $f(x) = \frac{x^2}{x-1}$

b) $f(x) = \frac{x}{x-1}$ e) $f(x) = \frac{3x}{x^2-1}$ h) $f(x) = \frac{x^2+2x-1}{x-1}$

c) $f(x) = \frac{x-1}{x^2+1}$ f) $f(x) = \frac{2x}{(x-1)^2}$ i) $f(x) = \frac{x^2}{-x^3+6x-4}$

Exercise 8: Tangent and normal lines
Find the formulae of the tangent line t(x) and the normal line n(x) to the graph of f in B:

a) $f(x) = \frac{1}{x^2}$ in B(1|?) d) $f(x) = \ln(x + 1)$ in B(0|?)

b) $f(x) = \sqrt{2x+1}$ in B(4|?) e) $f(x) = (2x - 1)\cdot\ln(x)$ in B(e|?)

c) $f(x) = \sin(3x)$ in B($\frac{\pi}{3}$|?) f) $f(x) = \frac{2x}{x+2}$ in B(2|?)

Exercise 9: Tangent lines with given gradient

Find the formulae and the contact points of all possible tangent lines with given gradient m to the graph of f

a) $f(x) = 2x^3 - 1$ with m = 6

b) $f(x) = x\cdot\sqrt{x}$ with m = 3

c) $f(x) = 4x + 3 + \frac{3}{x}$ with m = 1

d) $f(x) = \sin(2x)$ with m = 2 in the interval $[0; \infty]$

e) $f(x) = e^{2x-1}$ with m = 2e

f) $f(x) = \ln(2x - 1)$ with m = 2

Exercise 10: Tangent lines through given points

Find the formulae of the tangent lines to the graph of f throuh the given point.

a) $f(x) = e^x$ throuh $P(-1|-\ln(4))$

b) $f(x) = \frac{6}{x^2}$ throuh $P(\frac{3}{2}|0)$

c) $f(x) = \ln(x)$ throuh $P(-1|0)$

d) $f(x) = x\cdot e^x$ throuh $P(\frac{1}{2}-\frac{1}{2e^2}|-\frac{1}{e})$

e) $f(x) = \frac{3}{x^2+3}$ throuh $P(0|\frac{9}{8})$

f) $f(x) = \sin(\pi x) + 1$ throuh $P(\frac{1}{2}|1+\frac{\pi}{2})$

Exercise 11: Tangent and normal lines through given points

a) The tangent and normal lines to $f(x) = \frac{2}{x^2+1}$ in $P(-1|1)$ and $Q(1|1)$ form a rectangle. Find its area.

b) The tangent and normal lines to $f(x) = 2\cdot\cos(\frac{x}{2}) - 1$ in its inflection points define a quadrangle. Find its area.

c) A light ray along the vertical x = 3 is reflected at the silver nose of an airplane given by the curve $y = 2\sqrt{x}$. Where does it meet the y-axis? Hint: According to the law of reflection the incoming and the outgoing rays both have the same angle with respect to the normal line through the reflection point. Start with a drawing of the situation.

d) Two radio waves along the horizontals $y = 6\sqrt{3}$ and $y = 18\sqrt{3}$ are reflected at a satellite dish defined by the curve $y = 6\sqrt{x}$. The receiver is placed at the intersection point of the reflected waves. Find the coordinates of the receiver.

Exercise 12: Families of functions

a) Find the formula of the tangent line to $f_t(x) = e - e^{tx}$ through its x-intercept..

b) Find the formulae of all possible tangent lines through $P(3t - t^2|0)$ to the graphs of $f_t(x) = -\frac{1}{x-t} + 1$.

c) Find the y-intercept of the straight line connecting the two onflexion points of $f_t(x) = (x^2 + 2 - t^2)e^x$.

d) Find the area of the rhombus whose sides are given by the tangent lines through the outer inflexion points of $f_t(x) = x\cdot e^{-tx^2}$ and their mirror images with respect to the y-axis.

5.4. Solutions to the exercises on differentiation rules

Exercise 1: Compound functions
evident

Exercise 2: Chain rule

a) $f'(x) = 2x\cdot 3(x^2 + 1)^2$

b) $f'(x) = 3\cdot(4x + 3)\cdot(2x^2 + 3x - 1)^2$

c) $f'(x) = \frac{3}{2\sqrt{3x-1}}$

d) $f'(x) = \frac{4x+1}{2\sqrt{2x^2+x-3}}$

e) $f'(x) = -\frac{1}{(x-1)^2}$

f) $f'(x) = -\frac{2x}{(x^2-1)^2}$

g) $f'(x) = \frac{3x^2-6}{(-x^3+6x-4)^2}$

h) $f'(x) = (2x - 3)\cdot\cos(x^2 - 3x)$

i) $f'(x) = -3x^2\cdot\sin(x^3 + 1)$

Exercise 3: Chain rule with exponential functions

a) $f'(x) = \ln(2)\cdot 2^x$ b) $f'(x) = -\ln(10)\cdot 10^{-x}$ c) $f'(x) = 2x\cdot e^{x^2-1}$ d) $f'(x) = 3\cdot e^{3x-1}$ e) $f'(x) = -2x\cdot e^{-x^2}$

Exercise 4: Product rule

a) $f'(x) = (1 + x)\cdot e^x$

b) $f'(x) = (3x^2 + 7x - 1)\cdot e^x$

c) $f'(x) = 2x\cdot\sin(x) + x^2\cdot\cos(x)$

d) $f'(x) = 2x\cdot\sin(x) + (x^2 - 2)\cdot\cos(x)$

e) $f'(x) = 3x^2\cdot\cos(x) - x^3\cdot\sin(x)$

f) $f'(x) = (3x^2 - 2)\cdot\cos(x) - (x^3 - 2x + 1)\cdot\sin(x)$

Exercise 5: Chain rule and product rule

a) $f'(x) = (1-x)\cdot e^{-x}$

b) $f'(x) = (-x^2 + 2x + 2)\cdot e^{-x}$

c) $f'(x) = (2x^2+2x-4)\cdot e^{2x-1}$

d) $f'(x) = (2x^2 + 1)\cdot e^{x^2-1}$

e) $f'(x) = (2\cdot\ln(3)\cdot x^2+4x+3\cdot\ln(3))\cdot 3^x$

f) $f'(x) = (1 - x\cdot\ln(2))\cdot 2^{-x}$

g) $f'(x) = \frac{15x^2-4x-6}{2\sqrt{3x-1}}$

h) $f'(x) = -\frac{1}{(x-1)^2}$

Exercise 6: Chain rule and product rule with logarithm functions

a) $f'(x) = \frac{2}{2x-3}$

b) $f'(x) = \ln(x^2 + 3) + \frac{2x^2}{x^2+3}$

c) $f'(x) = 2\ln(x + 1) + \frac{2x-1}{x+1}$

d) $f'(x) = 2(\ln(x) + 1 - \frac{1}{x})$

e) $f'(x) = 1\cdot\ln(x) + x\cdot\frac{1}{x} - 1 = \ln(x)$

f) $f'(x) = \frac{3}{(3x+1)\cdot\ln(2)}$

Exercise 7: Quotient rule

a) $f'(x) = \frac{2x^2+6x-2}{(2x+3)^2}$

b) $f'(x) = -\frac{1}{(x-1)^2}$

c) $f'(x) = -\frac{x^2-2x-1}{(x^2+1)^2}$

d) $f'(x) = \frac{2x^2+2x-2}{(2x+1)^2}$

e) $f'(x) = -\frac{3(x^2+1)}{(x^2-1)^2}$

f) $f'(x) = -\frac{2(x+1)}{(x-1)^3}$

g) $f'(x) = \frac{x(x-2)}{(x-1)^2}$

h) $f'(x) = \frac{x^2-2x-1}{(x-1)^2}$

i) $f'(x) = \frac{x^4+6x^2-8x}{(-x^3+6x-4)^2}$

Exercise 8: Tangent and normal lines

a) $B(1|1)$ with $t(x) = -2x + 3$ and $n(x) = \frac{1}{2}x + \frac{1}{2}$

b) $B(4|3)$ with $t(x) = \frac{1}{3}x + \frac{5}{3}$ and $n(x) = -3x + 15$

c) $B(\frac{\pi}{3}|0)$ with $t(x) = 3x + \pi$ and $n(x) = \frac{1}{3}x - \frac{\pi}{9}$

d) $B(0|0)$ with $t(x) = x$ and $n(x) = -x$

e) $B(e|2e - 1)$ with $t(x) = \frac{4e-1}{e}x - 2e$
and $n(x) = \frac{-e}{4e-1}x + \frac{e^2}{4e-1} + 2e - 1$

f) $B(2|1)$ with $t(x) = \frac{1}{4}x + \frac{1}{2}$ and $n(x) = -4x + 9$

Exercise 9: Tangent lines with given gradient

a) $t_1(x) = 6x + 3$ in $B_1(-1|-3)$ and
$t_2(x) = 6x - 5$ in $B_2(1|10)$

d) $t_1(x) = 2x$ in $B_1(0|0)$ and
$t_2(x) = 2x - 2\pi$ in $B_2(\pi|0)$

b) $t(x) = 3x - 4$ in B(4|8)
c) $t_1(x) = x - 3$ in $B_1(-1|-4)$ and $t_2(x) = x + 9$ in $B_2(1|10)$
e) $t(x) = 2ex - e$ in B(1|e)
f) $t(x) = 2x - 2$ in B(1|0)

Exercise 10: Tangent lines through given points

a) $t(x) = 2x + 2 - \ln(4)$ with B(ln(2)|2)
b) $t(x) = -12x + 18$ with B(1|6)
c) $t(x) = x - 1$ with B(1|0)
d) $t_1(x) = -\frac{1}{e}$ with $B_1(-1|-\frac{1}{e})$ (min!) and $t_2(x) = 2ex - e$ with $B_2(1|e)$.
e) $t_{1/2}(x) = \pm\frac{3}{8}x + \frac{9}{8}$ with $B_{1/2}(\pm 1|\frac{3}{4})$ (inflexion points!)
f) $t_1(x) = \pi x + 1$ through $B_1(0|1)$ und $t_2(x) = -\pi x + \pi + 1$ with $B_2(1|1)$ (inflexion point!)

Exercise 11: Tangent and normal lines through given Punpointskte

a) The tangent lines $t_{1/2}(x) = \pm x + 2$ meet in $S_t(0|2)$ and the normal lines $n_{1/2}(x) = \pm x$ in $S_n(0|0)$. The Quadrangle PS_nQS_t has two diagonals with length 2. Thus we have a square with sidelengths $\sqrt{2}$ LE and area 2 FE.

b) The inflexion points are $Inf_{1/2}(\pm\frac{\pi}{2}|0)$. The tangent lines $t_{1/2}(x) = \pm x + \frac{\pi}{2}$ meet in $S_t(0|\frac{\pi}{2})$ and the normal lines $n_{1/2}(x) = \pm x - \frac{\pi}{2}$ in $S_n(0|-\frac{\pi}{2})$. The quadrangle PS_nQS_n has two diagonals with length π. Thus it is a squre with sidlengths $\frac{\pi}{\sqrt{2}}$ and area $\frac{\pi^2}{2}$.

c) The light ray hits the nose in $P(3|2\sqrt{3})$. Th gradient in this point is $a = \frac{1}{\sqrt{3}} = \tan(\alpha)$ with elevation angle $\alpha = \tan^{-1}(\frac{1}{\sqrt{3}}) = 30°$. The angle to the normal line (=incident angle) is $90° - 30° = 60°$. The reflection angle is $30° - 60° = -30°$. So the gradient of the reflected ray is $a' = \tan(-30°) = -\frac{1}{\sqrt{3}}$. Its formula is $s'(x) = -\frac{1}{\sqrt{3}}x + 5\sqrt{3}$ with y-intercept $S(0|5\sqrt{3})$.

d) The radio waves hit the dish in $P_1(3|6\sqrt{3})$ and $P_2(27|18\sqrt{3})$. The corresponding gradients are $a_1 = \sqrt{3} = \tan(\alpha_1)$ with elevation angle $\alpha_1 = \tan^{-1}(\sqrt{3}) = 60°$ and $a_2 = \frac{1}{\sqrt{3}} = \tan(\alpha_2)$ with $\alpha_2 = \tan^{-1}(\frac{1}{\sqrt{3}}) = 30°$. The incident angle resp. reflection angle with respect to the normal line therefore are 30° und 60°. The elevation angles of the reflected waves are $\alpha_1' = -60°$ and $\alpha_2' = 60°$ (refer to drawing!). The resulting gradients are $a_1' = \tan(-60°) = -\sqrt{3}$ and $a_2' = \tan(60°) = \sqrt{3}$. Their formulae are $s_1'(x) = -\sqrt{3}x + 9\sqrt{3}$ und $s_2'(x) = \sqrt{3}x - 9\sqrt{3}$. They meet in S(9|0). this must be the location of the receiver.

Exercise 12: Families of functions

a) $f_t(x) = e - e^{tx}$ with $f_t'(x) = -te^{tx}$ has the tangent line $t(x) = -tx + 1$ in $S_t(\frac{1}{t}|0)$.

b) $f_t(x) = -\frac{1}{x-t} + 1$ with $f_t'(x) = \frac{1}{(x-t)^2}$ has the tangent line $g_t(x) = \frac{x}{(2-t)^2} + \frac{t^2-3t}{(2-t)^2}$ in $B_t(2|\frac{1-t}{2-t})$ and $h_t(x) = \frac{x}{t^2} + \frac{t-3}{t}$ in $B_t(2t|\frac{t-1}{t})$

c) $f_t(x) = (x^2 + 2 - t^2)e^x$ with $f_t'(x) = (x^2 - 2x + 2 - t^2)e^x$ and $f_t''(x) = (x^2 - t^2)e^x$ has the inflexion points $Inf_{1t}(-t|\frac{2}{e^t})$ and $Inf_{2t}(t|2e^t)$ with connecting line $g_t(x) = \frac{1}{t}(e^t - e^{-t})x + e^t + e^{-t}$ with y-intercept $S_{ty}(0|e^t + e^{-t})$.

d) $f_t(x) = x\cdot e^{-tx^2}$ with $f_t'(x) = (-2tx^2 + 1)\, e^{-tx^2}$ and $f_t''(x) = (4t^2 x^3 - 6tx)\, e^{-tx^2}$ has the inflexion points $Inf_{t0}(0|0)$ und $Inf_{t1/2}(\pm\sqrt{\frac{3}{2t}}|\sqrt{\frac{3}{2t}}\, e^{-1,5})$ with tangent lines $t_o(x) = x$ and $t_{1/2}(x) = -2e^{-1,5}x \pm \frac{1}{2}$. So the rhombus has the area $A = \sqrt{\frac{3}{2t}}$ FE.