5.4. Exercises on rational functions

Exercise 1: Zeroes of n^{th} Order

Find the asymptotes and intercepts of the following functions and draw their graphs. Check the results with your GDC

a) $f(x) = (x-2)(x+1)$ b) $f(x) = (x-2)(x+1)^2$ c) $f(x) = (x+2)^2(x-1)^2$ d) $f(x) = (x-2)x^2(x+1)$

e) $f(x) = \frac{1}{x+3}$ f) $f(x) = \frac{1}{(x-1)^2}$ g) $f(x) = \frac{x+1}{(x-1)^2}$ h) $f(x) = \frac{x+1}{(x-2)(x+3)}$

i) $f(x) = \frac{(x+1)^2}{(x-2)(x+3)^2}$ j) $f(x) = \frac{(x-4)(x+1)}{x(x+3)^2}$ k) $f(x) = \frac{(x-4)(x+1)}{x^2(x+3)}$ l) $f(x) = \frac{(x-4)^2}{x^2(x+3)}$

Exercise 2: Zeroes of n^{th} Order

Give the formula of a function with the following properties and check the result with your GDC

a) a zero with sign change but no saddle point at $x = 2$
b) a saddle point in $S(-1|0)$
c) a maximum in $Max(-3|0)$
d) a minimum in $Min(1|0)$
e) a minimum in $Min(2|0)$ and a sign change at $x = 4$
f) a maximum in $Max(-3|0)$, a sign change at $x = 0$ and a minimum in $Min(2|0)$
g) a saddle point in $S(-1|0)$, a maximum in $Max(3|0)$ and a minimum in $Min(-5|0)$
h) a vertical asymptote with sign change at $x = 2$
i) a vertical asymptote without sign change at $x = 2$ and y-intercept $S_y(0|-1)$
j) a vertical asymptote with sign change at $x = -2$ and a zero with sign change at $x = 2$
k) two vertical asymptotes with sign change at $x = 1$ and $x = 3$ and a Minimum in $Min(5|0)$
l) two vertical asymptotes with sign change at $x = 1$ and $x = 3$ and a Maximum in $Max(5|0)$
m) a vertical asymptote with sign change at $x = -3$, another one without sign change at $x = 2$ and a maximum in $Max(1|0)$
n) two vertical asymptotes without sign change at $x = -2$ and $x = 4$ and in the middle between them a saddle point.

Exercise 3: Zeroes of n^{th} Order with factorization

Find the asymptotes and intercepts of the following functions and draw their graphs.

a) $f(x) = \frac{1}{x^2-x-2}$ b) $f(x) = \frac{1}{4-x^2}$ c) $f(x) = \frac{1}{x^2+4x+4}$ d) $f(x) = \frac{1}{x^2+1}$

e) $f(x) = \frac{x+1}{x^2-4}$ f) $f(x) = \frac{x}{x^2-6x+9}$ g) $f(x) = \frac{x^2}{x^3-1}$ h) $f(x) = \frac{x-1}{x^3+4x^2+3x}$

Exercise 4: Holes

Find the domains, holes, asymptotes and intercepts of the following functions and draw their graphs.

a) $f(x) = \frac{x+3}{x+3}$ b) $f(x) = \frac{x+1}{x^2-x-2}$ c) $f(x) = \frac{x^2+2x-3}{x+3}$ d) $f(x) = \frac{x^2+2x+1}{x^2-3x-4}$

e) $f(x) = \frac{x+1}{x^2-1}$ f) $f(x) = \frac{x}{x^3-4x^2+4x}$ g) $f(x) = \frac{x^2-1}{x^3-1}$ h) $f(x) = \frac{x-1}{x^3-x^2+x-1}$

Exercise 5: Vertical shift

Write the following functions as a single fraction. Find their asymptotes and intercepts and draw their graphs.

a) $f(x) = \frac{1}{x-1} + 1$ b) $f(x) = \frac{1}{x+4} + 3$ c) $f(x) = \frac{3}{x+1} - 2$ d) $f(x) = \frac{1}{(x+2)^2} + 2$

e) $f(x) = \frac{1}{(x-4)^2} - 1$ f) $f(x) = -\frac{1}{(x+1)^2} + 1$ g) $f(x) = \frac{x}{(x-1)^2} - 2$ h) $f(x) = \frac{x}{x^2-1} + 1$

Exercise 6: Vertical shift

Write these functions in the form $f(x) = \frac{a}{g(x)} + b$. Find their asymptotes, holes and intercepts and draw their graphs.

a) $f(x) = \frac{x-3}{x+3}$ b) $f(x) = \frac{2x+2}{x+3}$ c) $f(x) = \frac{-x+3}{x+2}$ d) $f(x) = \frac{x^2+2x+1}{2x^2+8x+6}$

e) $f(x) = \frac{3x-6}{x+3}$ f) $f(x) = \frac{4-x}{x+1}$ g) $f(x) = \frac{x^2+4x+4}{x^2-4x+4}$ h) $f(x) = \frac{2x^2-4x+2}{x^2+4x+3}$

Exercise 7: Asymptotic lines
Find the asymptotes, domains, holes and intercepts of the following functions and draw their graphs.

a) $f(x) = \dfrac{x^2}{x-3}$ b) $f(x) = \dfrac{-x^2+1}{x+2}$ c) $f(x) = \dfrac{x^2-2x-3}{2x-2}$ d) $f(x) = \dfrac{x^3-2x^2+x}{x^2+2x+1}$

Exercise 8: Families of functions

Find the asymptotes, domains, holes and intercepts of the following functions and draw their graphs for $t \in \{-2; 0; 2\}$

a) $f_t(x) = \dfrac{1}{x-t}$ b) $f_t(x) = \dfrac{x-t}{x}$ c) $f_t(x) = \dfrac{1}{x^2-t}$ d) $f_t(x) = \dfrac{x+t}{x^2-2tx+t^2}$ e) $f_t(x) = \dfrac{x^2+2tx+t^2}{tx-t^2}$

Exercise 9: Finding functions with given properties
Give the formula of a simple rational function with the following properties:
a) A zero at $x = 1$ and a vertical asymptote with sign change at $x = 3$.
b) A zero at $x = 1$ and a vertical asymptote without sign change at $x = 3$.
c) A zero at $x = 0$ and two vertical asymptotes with sign change at $x = -1$ and $x = 1$.
d) Intercepts $A(-1|0)$ and $B(0|1)$ and a vertical asymptote with sign change at $x = -4$.
e) A horizontal asymptote at $y = 2$, a vertical asymptote with sign change at $x = 1$ and a zero at $x = -5$.
f) An asymptotic line $y = x + 1$ and a vertical asymptote with sign change at $x = -2$.
g) An asymptotic line $y = -\frac{1}{2}x + 1$, a vertical asymptote with sign change at $x = 1$ und a zero at $x = 3$.
h) An asymptotic curve $y = x^2 - 1$ and two vertical asymptotes without sign change $x = -1$ and $x = 1$.

Exercise 10: Curve analysis of rational functions
Examine the function f with regard to intercepts, extrema, inflexion points, behaviour for $x \to \pm\infty$ and draw its graph.

a) $f(x) = \dfrac{4x}{1-x^2}$ b) $f(x) = \dfrac{(x+1)^2}{(x-1)^2}$ c) $f(x) = \dfrac{x^2-1}{x^2+1}$ d) $f(x) = \dfrac{x^2-7x+11}{x-5}$

Exercise 11: Curve analysis of families of rational functions
Examine the functions f_t for $t > 0$ with regard to intercepts, behaviour for $x \to \pm\infty$, extrema, inflexion points and draw their graphs for $t \in \{-2; 0; 2\}$.
Find the loci of all extrema and inflexion points.

a) $f_t(x) = x + \dfrac{4t^3}{(x-t)^2}$ c) $f_t(x) = \dfrac{x-t}{x^2}$ e) $f_t(x) = \dfrac{x+t}{(x-t)^2}$

b) $f_t(x) = x + t + \dfrac{1}{x-t}$ d) $f_t(x) = \dfrac{x}{(x-t)^2}$ f) $f_t(x) = x + 3t + \dfrac{4t^2}{x-t}$

Exercise 12: Optimization problems with rational functions
a) Which rectangle with area $A = 36\ cm^2$ has the smallest circumference??
b) Which sector with area $A = 100\ cm^2$ has the smallest circumference?
c) Which point of the graph of $f(x) = \dfrac{2}{x^2}$ has the smallest distance to the origin? **Hint**: Because of the monotonicity of the root function $\sqrt{A}$ is minimal if and only if A is minimal. So it suffices to examine $x^2 + y^2$.
d) Which rectangle between the coordinate axes and the graph of $f(x) = \dfrac{20}{x^2+5}$ has the largest area?
e) The cross section of an underground sewer duct is a rectangle with a semicircle as roof and is to have an area of $8\ m^2$ Find its dimensions so that the number of bricks needed for the wall and roof is minimal.
f) Find the dimensions of a cylindrical 1 Litre tin can with minimal use of tin.
g) Find the dimensions of a 1 Litre tin can which consists of a cylindrical main body with hemispherical top an flat bottom so that the material use is minimal.
h) Find the dimensions of a cuboid tin container with quadratic cross section and minimal material use.

4.6. Solutions to the exercises on rational functions

Exercise 1: Zeroes of n^{th} Order

a) $S_y(0|-2)$; $S_{x1}(-1|0)$; $S_{x2}(2|0)$, no asymptotes

b) $S_y(0|-2)$; $S_{x1}(-1|0)$ (double ⇒ extremum); $S_{x2}(2|0)$, no asymptotes

c) $S_y(0|-2)$; $S_{x1}(-2|0)$ (double ⇒ extremum); $S_{x2}(1|0)$ (double ⇒ contact point), no asymptotes

d) S(0|0); (double ⇒ extremum); $S_{x1}(-2|0)$; $S_{x2}(1|0)$, no asymptotes

e) $S_y(0|\frac{1}{3})$, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptote with sign change at x = −3 because of single zero in the denominator.

f) $S_y(0|1)$, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptote without sign change at x = 1 because of double zero in the denominator.

g) $S_y(0|1)$, $S_x(-1|0)$; horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptote without sign change at x = 1 because of double zero in the denominator.

h) $S_y(0|-\frac{1}{6})$, $S_x(-1|0)$; horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptotes with sign change at x = −3 and x = 2 because of single zeroes in the denominator.

i) $S_y(0|-\frac{1}{6})$, $S_x(-1|0)$ (double ⇒ extremum); horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptotes without sign change at x = −3 (double zero in denominator) and with sign change at x = 2 (single zero in denominator)

j) $S_{x1}(-1|0)$; $S_{x2}(4|0)$; horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptotes without sign change at x = −3 (double zero in denominator) and with sign change at x = 0 (single zero in denominator)

k) $S_{x1}(-1|0)$; $S_{x2}(4|0)$; horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptotes without sign change at x = 0 (double zero in denominator) and with sign change at x = −3 (single zero in denominator)

l) $S_x(4|0)$ (double ⇒ extremum); horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$ and vertical asymptotes without sign change at x = 0 (double zero in denominator) and with sign change at x = −3 (single zero in denominator)

Exercise 2: Zeroes of n^{th} Order

a) $f(x) = x - 2$
b) $f(x) = (x + 1)^3$
c) $f(x) = -(x + 3)^2$
d) $f(x) = (x - 1)^2$
e) $f(x) = -(x - 2)^2(x - 4)$
f) $f(x) = (x + 3)^2x(x - 2)^2$
g) $f(x) = -(x + 5)^2(x - 1)^3(x - 3)^2$
h) $f(x) = \frac{1}{x-2}$
i) $f(x) = -\frac{4}{(x-2)^2}$
j) $f(x) = \frac{x-2}{x+2}$
k) $f(x) = \frac{(x-5)^2}{(x-1)(x-3)}$
l) $f(x) = -\frac{(x-5)^2}{(x-1)(x-3)}$
m) $f(x) = -\frac{(x-1)^2}{(x+3)(x-2)^2}$
n) $f(x) = \frac{(x-1)^3}{(x+2)^2(x-4)^2}$

Exercise 3: Zeroes of n^{th} Order with factorization

a) $f(x) = \frac{1}{(x-2)(x+1)}$ ⇒ D = $\mathbb{R}\backslash\{-1; 2\}$, $S_y(0|-\frac{1}{2})$, vertical asymptote with sign change at x = −1 and 2 because of single zeroes in the denominator, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

a) $f(x) = -\frac{1}{(x-2)(x+2)}$ ⇒ D = $\mathbb{R}\backslash\{\pm 2\}$, $S_y(0|\frac{1}{4})$, vertical asymptote with sign change at x = ±2 because of single zero in the denominator, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

b) $f(x) = \frac{1}{(x+2)^2}$ ⇒ D = $\mathbb{R}\backslash\{-2\}$, $S_y(0|\frac{1}{4})$, vertical asymptote without sign change at x = −2 because of double zero in the denominator, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

c) $f(x) = \frac{1}{x^2+1}$ ⇒ D = $\mathbb{R}$, $S_y(0|1)$, no vertical asymptote because no zero in the denominator, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

d) $f(x) = \frac{x+1}{(x-2)(x+2)} \Rightarrow D = \mathbb{R}\setminus\{\pm 2\}$, $S_y(0|-\frac{1}{4})$, $S_x(-1|0)$ with sign change because of single zero in the numerator, vertical asymptote with sign change at x = ±2 because of single zeroes in the denominator, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

e) $f(x) = \frac{x}{(x-3)^2} \Rightarrow D = \mathbb{R}\setminus\{3\}$, $S_x(0|0)$ with sign change because of single zero in the numerator, vertical asymptote without sign change at x = 3 because of double zero in the denominator, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

f) $f(x) = \frac{x-4}{x^2+2} \Rightarrow D = \mathbb{R}$, $S_y(0|-2)$ $S_x(4|0)$ with sign change because of single zero in the numerator, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

g) $f(x) = \frac{x^2}{(x-1)(x^2+x+1)} \Rightarrow D = \mathbb{R}\setminus\{1\}$; S(0|0) without sign change because of double zero in the numerator, vertical asymptote with sign change at x = 1, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

h) $f(x) = \frac{x-1}{x(x+1)(x+3)} \Rightarrow D = \mathbb{R}\setminus\{-3;-1;0\}$; $S_x(1|0)$ with sign change because of single zero in the numerator, vertical asymptotes with sign change at x = 0; −1; −3, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

Exercise 4: Holes

a) $f(x) = \frac{x+3}{x+3} = 1$ with Domain = $\mathbb{R}\setminus\{-3\}$, $S_y(0|1)$, Hole H(−3|1), horizontal asymptote y = 1 since $\lim_{x\to\pm\infty} f(x) = 1$

b) $f(x) = \frac{x+1}{(x+1)(x-2)} = \frac{1}{x-2}$ with Domain = $\mathbb{R}\setminus\{-1;2\}$, $S_y(0|-\frac{1}{2})$, Hole $H(-1|-\frac{1}{3})$, vertical asymptote with sign change because of single zero in the denominator at x = 2, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$

c) $f(x) = \frac{(x+3)(x-1)}{(x+3)} = x - 1$ with Domain = $\mathbb{R}\setminus\{-3\}$, $S_y(0|-1)$, Hole H(−3|−4), vertical asymptote with sign change because of single zero in the denominator at x = −3, asymptotic line g(x) = x − 1 since $\lim_{x\to\pm\infty} f(x) - g(x) = 0$

d) $f(x) = \frac{(x+1)^2}{(x+1)(x-4)} = \frac{x+1}{x-4}$ with Domain = $\mathbb{R}\setminus\{-1;4\}$, $S_y(0|-\frac{1}{4})$, Hole H(−1|0), vertical asymptote with sign change because of single zero in the denominator at x = 4, horizontal asymptote y = 1 since $\lim_{x\to\pm\infty} f(x) = 1$

e) $f(x) = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}$ with Domain = $\mathbb{R}\setminus\{\pm 1\}$, $S_y(0|-1)$, Hole $H(-1|-\frac{1}{2})$, vertical asymptote with sign change because of single zero in the denominator at x = 1, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$.

f) $f(x) = \frac{x}{x(x-2)^2} = \frac{1}{(x-2)^2}$ with Domain = $\mathbb{R}\setminus\{0; 2\}$, Hole $L(0|\frac{1}{4})$, vertical asymptote without sign change because of double zero in the denominator at x = 2, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$.

g) $f(x) = \frac{x-1}{(x^2+1)(x-1)} = \frac{1}{x^2+1}$ with Domain = $\mathbb{R}\setminus\{1\}$, $S_y(0|1)$, Hole $L(1|\frac{1}{2})$, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$.

h) $f(x) = \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \frac{x+1}{x^2+x+1}$ with Domain = $\mathbb{R}\setminus\{1\}$, $S_y(0|1)$, $S_x(-1|0)$ with sign change because of single zero in the numerator, Hole $L(1|\frac{2}{3})$, horizontal asymptote y = 0 since $\lim_{x\to\pm\infty} f(x) = 0$.

Exercise 5: vertical shift

a) $f(x) = \frac{x}{x-1}$ $\Rightarrow$ S(0|0), vertical asymptote with sign change at x = 1 because of single zero in the denominator and horizontal asymptote y = 1 since $\lim_{x\to\pm\infty} f(x) = 1$

b) $f(x) = \frac{3x+13}{x+4}$ $\Rightarrow$ $S_y(0|\frac{13}{4})$, $S_y(-\frac{13}{3}|0)$, vertical asymptote with sign change at x = −4 and horizontal asymptote y = 3 since $\lim_{x\to\pm\infty} f(x) = 3$

c) $f(x) = \frac{1-2x}{x+1}$ $\Rightarrow$ $S_y(0|1)$, $S_x(\frac{1}{2}|0)$, vertical asymptote with sign change at x = −1 and horizontal asymptote y = −2 since: $\lim_{x\to\pm\infty} f(x) = -2$

d) $f(x) = \frac{x^2+4x+5}{x^2+4x+4}$ $\Rightarrow$ $S_y(0|\frac{5}{4})$, vertical asymptote without sign change at x = −2 and horizontal asymptote y = 2 since $\lim_{x\to\pm\infty} f(x) = 2$

e) $f(x) = \frac{-x^2+8x-15}{x^2-8x+16}$ $\Rightarrow$ $S_y(0|-\frac{15}{16})$, $S_{x1/2}(4\pm1|0)$, vertical asymptote without sign change at x = 4 and horizontal asymptote y = −1 since $\lim_{x\to\pm\infty} f(x) = -1$

f) $f(x) = -\frac{x^2+2x}{x^2+2x+1}$ $\Rightarrow$ $S_{x1/2}(-1\pm1|0)$, vertical asymptote without sign change at x = −1 and horizontal asymptote y = 1 since $\lim_{x\to\pm\infty} f(x) = 1$

g) $f(x) = \frac{-2x^2+5x-2}{x^2-2x+1}$ $\Rightarrow$ $S_y(0|-2)$, $S_{x1/2}(-\frac{5}{4}\pm\frac{3}{4}|0)$, vertical asymptote without sign change at x = 1 and horizontal asymptote y = −2 since $\lim_{x\to\pm\infty} f(x) = -2$

h) $f(x) = \frac{x^2+x-1}{x^2-1}$ $\Rightarrow$ $S_y(0|1)$, $S_{x1/2}(\frac{1}{2}\pm\frac{\sqrt{5}}{2}|0)$, vertical asymptotes with sign change at x = ±1 and horizontal asymptote y = 1 since $\lim_{x\to\pm\infty} f(x) = 1$

Exercise 6: Vertical shift

a) $f(x) = \frac{x-3}{x+3} = 1 - \frac{6}{x+3}$ $\Rightarrow$ D = $\mathbb{R}\setminus\{-3\}$, $S_y(0|-1)$, $S_x(3|0)$ with sign change, vertical asymptote with sign change at x = −3, horizontal asymptote y = 1

b) $f(x) = \frac{2x+2}{x+3} = 2 - \frac{4}{x+3}$ $\Rightarrow$ D = $\mathbb{R}\setminus\{-3\}$, $S_y(0|\frac{2}{3})$, $S_x(-1|0)$ with sign change, vertical asymptote with sign change at x = −3, horizontal asymptote y = 2

c) $f(x) = \frac{-x+3}{x+2} = -1 + \frac{5}{x+2}$ $\Rightarrow$ D = $\mathbb{R}\setminus\{-2\}$, $S_y(0|\frac{3}{2})$, $S_x(3|0)$ with sign change, vertical asymptote with sign change at x = −2, horizontal asymptote y = −1

d) $f(x) = \frac{(x+1)^2}{2(x+1)(x+3)} = \frac{x+1}{2x+6} = \frac{1}{2} - \frac{2}{x+3}$ $\Rightarrow$ D = $\mathbb{R}\setminus\{-1; -3\}$, $S_y(0|\frac{1}{6})$, Hole H(1|0) with sign change, vertical asymptote with sign change at x = −3, horizontal asymptote y = $\frac{1}{2}$

e) $f(x) = \frac{2(x-2)}{x+3} = 2 - \frac{10}{x+3}$ $\Rightarrow$ D = $\mathbb{R}\setminus\{-3\}$, $S_y(0|-\frac{4}{3})$, $S_x(2|0)$ with sign change, vertical asymptotes with sign change at x = −3, horizontal asymptote y = 2

f) $f(x) = \frac{-(x-4)}{x+1} = -1 + \frac{5}{x+1}$ $\Rightarrow$ D = $\mathbb{R}\setminus\{-1\}$, $S_y(0|4)$, $S_x(4|0)$ with sign change, vertical asymptotes with sign change at x = −1, horizontal asymptote y = −1

g) $f(x) = \frac{(x+2)^2}{(x-2)^2} = 1 + \frac{8x}{x^2-4x+4}$ $\Rightarrow$ D = $\mathbb{R}\setminus\{2\}$, $S_y(0|-1)$, $S_x(-2|0)$ without sign change, vertical asymptotes without sign change at x = 2, horizontal asymptote y = 1

h) $f(x) = \frac{2(x-1)^2}{(x+1)(x+3)} = 2 - \frac{12x+4}{x^2+4x+3} \Rightarrow D = \mathbb{R}\backslash\{-1; -3\}$, $S_y(0|\frac{2}{3})$, $S_x(1|0)$ without sign change, vertical asymptotes with sign change at x = −1 and x = −3, horizontal asymptote y = 2

Exercise 7: Asymptotic lines

a) $f(x) = \frac{x^2}{x-3} = x + 3 + \frac{9}{x-3} \Rightarrow D = \mathbb{R}\backslash\{3\}$, $S_x(0|0)$ without sign change, vertical asymptotes with sign change at x = 3, asymptotic line y = x + 3.

b) $f(x) = \frac{-(x-1)(x+1)}{x+2} = -x + 2 - \frac{3}{x+2} \Rightarrow D = \mathbb{R}\backslash\{-2\}$, $S_y(0|\frac{1}{2})$, $S_{x1}(-1|0)$ and $S_{x2}(1|0)$ with sign change, vertical asymptotes with sign change at x = −2, asymptotic line y = −x + 2

c) $f(x) = \frac{1}{2}\frac{(x+1)(x-3)}{x-1} = \frac{1}{2}x - \frac{1}{2} - \frac{4}{2x-2} \Rightarrow D = \mathbb{R}\backslash\{1\}$, $S_y(0|\frac{3}{2})$, $S_{x1}(-1|0)$ and $S_{x2}(3|0)$ with sign change, vertical asymptotes with sign change at x = 1, asymptotic line $y = \frac{1}{2}x - \frac{1}{2}$

d) $f(x) = \frac{x(x-1)^2}{(x+1)^2} = x - 4 + \frac{8x+4}{x^2+2x+1} \Rightarrow D = \mathbb{R}\backslash\{-1\}$, S(0|0), $S_{x2}(1|0)$ without sign change, vertical asymptotes without sign change at x = −1, asymptotic line y = x − 4

Exercise 8: Families of functions

a) $f_t(x) = \frac{1}{x-t} \Rightarrow D = \mathbb{R}\backslash\{t\}$ with $S_y(0|-\frac{1}{t})$, vertical asymptote with sign change at x = t and horizontal asymptote y = 0

b) $f_t(x) = 1 - \frac{t}{x} \Rightarrow D = \mathbb{R}\backslash\{0\}$ with $S_x(t|0)$ with sign change, vertical asymptote with sign change at x = 0 and horizontal asymptote y = 1

c) $f_t(x) = \frac{1}{(x-\sqrt{t})\cdot(x+\sqrt{t})} \Rightarrow D = \mathbb{R}\backslash\{\pm\sqrt{t}\}$ only for t ≥ 0 and D = ℝ for t < 0 with $S_y(0|-\frac{1}{t})$ only for t ≠ 0, vertical asymptote with sign change at $x = \pm\sqrt{t}$ only for t ≥ 0 and horizontal asymptote y = 0 for t ∈ ℝ

d) $f_t(x) = \frac{x+t}{(x-t)^2} \Rightarrow D = \mathbb{R}\backslash\{t\}$ with $S_y(0|\frac{1}{t})$, $S_x(-t|0)$ with sign change, vertical asymptote without sign change at x = t and horizontal asymptote y = 0

e) $f_t(x) = \frac{1}{t}\frac{(x+t)^2}{x-t} = \frac{1}{t}x + 3 + \frac{4t}{x-t} \Rightarrow D = \mathbb{R}\backslash\{t\}$ with $S_y(0|-1)$, $S_x(-t|0)$ without sign change, vertical asymptote with sign change at x = t and asymptotic line $y = \frac{1}{t}x + 3$.

Exercise 9: Finding functions with given properties (examples)

a) $f(x) = \frac{x-1}{x-3}$

b) $f(x) = \frac{x-1}{(x-3)^2} = \frac{x-1}{x^2-6x+9}$

c) $f(x) = \frac{x}{(x-1)(x+1)} = \frac{x}{x^2-1}$

d) $f(x) = \frac{4(x+1)}{x+4} = \frac{4x+4}{x+4}$

e) $f(x) = 2 + \frac{12}{x-1} = \frac{2x+10}{x-1}$

f) $f(x) = x + 1 + \frac{1}{x+2} = \frac{x^2+3x+3}{x+2}$

g) $f(x) = -\frac{1}{2}x + 1 + \frac{1}{x-1} = -\frac{x\cdot(x-3)}{2(x-1)} = \frac{-x^2+3x}{2x-2}$

h) $f(x) = x^1 - 1 + \frac{1}{(x^2-1)^2} = \frac{x^6-3x^4+3x^2}{x^4-2x^2+1}$

Exercise 10: Curve analysis of rational functions

a) $f(x) = \frac{4x}{1-x^2} = \frac{4x}{(x-1)(x+1)}$: Domain = ℝ\{−1;1}, point symmetry to origin since f(−x) = −f(x), intercept: S(0|0), vertical asymptotes at x = ±1 since zeroes only in denominator, horizontal asymptote y = 0 for x → ± ∞ since degree of denominator > degree of numerator. $f'(x) = \frac{4(1+x^2)}{(1-x^2)^2}$ and $f''(x) = \frac{8x(3+x^2)}{(1-x^2)^3} \Rightarrow$ Inf(0|0)

b) $f(x) = \frac{(x+1)^2}{(x-1)^2} = 1 + \frac{4x}{(x-1)^2}$: Domain = $\mathbb{R}\backslash\{1\}$, no symmetry since $f(-x) \neq \pm f(x)$, intercepts $S_y(0|1)$ and $S_x(-1|0)$, vertical asymptote without sign change at x = 1 since double zero only in the denominator, horizontal asymptote y = 1 for $x \to \pm\infty$ since $f(x) \to 1$ for $x \to \pm\infty$. $f'(x) = -\frac{4(x+1)}{(x-1)^3}$ and $f''(x) = \frac{8(x+2)}{(x-1)^4}$ $\Rightarrow$ Min(−1|0) and Inf(−2| $\frac{1}{9}$)

c) $f(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$: Domain = $\mathbb{R}$, symmetry to y-axis since $f(-x) = f(x)$, intercepts $S_y(0|-1)$ and $S_{x1/2}(\pm1|0)$, horizontal asymptote y = 1 for $x \to \pm\infty$ since $f(x) \to 1$ for $x \to \pm\infty$. $f'(x) = \frac{4x}{(x^2+1)^2}$ and $f''(x) = \frac{4-12x^2}{(x^2+1)^3}$ $\Rightarrow$ Min(0|−1) and $\mathrm{Inf}_{1/2}(\pm\frac{1}{\sqrt{3}}|-\frac{1}{2})$.

d) $f(x) = \frac{x^2-7x+11}{x-5} = x - 2 + \frac{1}{x-5}$, Domain = $\mathbb{R}\backslash\{5\}$, no symmetry, intercepts: $S_y(0|-\frac{11}{5})$ and $S_{x1/2}(\frac{7}{2} \pm \sqrt{\frac{5}{4}}|0)$, vertical asymptote at x = 5 since zero only in denominator, asymptotic line g(x) = x − 2 for $x \to \pm\infty$ since $f(x) - g(x) = \frac{2}{x^2+1} \to 0$ for $x \to \pm\infty$. $f'(x) = 1 - \frac{1}{(x-5)^2} = \frac{x^2-10x+24}{(x-5)^2}$ and $f''(x) = \frac{2}{(x-5)^3}$ $\Rightarrow$ Max(4|1) and Min(6|5)

Exercise 11: Curve analysis of families of rational functions

a) $f_t(x) = x + \frac{4t^3}{(x-t)^2} = \frac{x^3-2tx^2+t^2x+4t^3}{(x-t)^2}$ $\Rightarrow$ Domain = $\mathbb{R}\backslash\{t\}$, $S_y(0|4t)$, $S_x(t|0)$, vertical asymptote without sign change at x = t and asymptotic line y = x. $f_t'(x) = 1 - \frac{8t^3}{(x-t)^3}$ and $f_t''(x) = \frac{24t^3}{(x-t)^4}$ $\Rightarrow$ Max(3t|4t) with locus $y = \frac{4}{3}x$

b) $f_t(x) = x + t + \frac{1}{x-t} = \frac{x^2-t^2+1}{x-t}$ $\Rightarrow$ Domain = $\mathbb{R}\backslash\{t\}$ with $S_y(0|-\frac{1}{t})$, $S_x(\pm\sqrt{t^2-1}|0)$ without sign change only for $|t| \geq 1$, vertical asymptote with sign change at x = t and asymptotic line y = x + t. $f_t'(x) = 1 - \frac{1}{(x-t)^2}$ and $f_t''(x) = \frac{2}{(x-t)^3}$ $\Rightarrow$ Max(t − 1|2t − 2) with locus y = 2x and Min(t + 1|2t + 2) with the same locus y = 2x (!)

c) $f_t(x) = \frac{x-t}{x^2} = \frac{1}{x} - \frac{t}{x^2}$ $\Rightarrow$ Domain = $\mathbb{R}\backslash\{0\}$ with $S_x(t|0)$ with sign change, vertical asymptote without sign change at x = 0 and horizontal asymptote y = 0. $f_t'(x) = -\frac{1}{x^2} + \frac{2t}{x^3} = \frac{2t-x}{x^3}$ and $f_t''(x) = \frac{2}{x^3} - \frac{6t}{x^4} = \frac{2x-6t}{x^4}$ $\Rightarrow$ Max(2t| $\frac{1}{4t}$) with locus $y = \frac{1}{2x}$ and Inf(3t| $\frac{2}{9t^2}$) with locus $y = \frac{2}{x^2}$.

d) $f_t(x) = \frac{x}{(x-t)^2}$ $\Rightarrow$ Domain = $\mathbb{R}\backslash\{t\}$ with $S_y(0|0)$ with sign change, vertical asymptote without sign change at x = t and horizontal asymptote y = 0. $f_t'(x) = -\frac{x+t}{(x-t)^3}$ and $f_t''(x) = \frac{2x+4t}{(x-t)^4}$ $\Rightarrow$ Min(−t|− $\frac{1}{4t}$) with locus $y = \frac{1}{4x}$ and Inf(−2t|− $\frac{2}{9t}$) with locus $y = \frac{4}{9x}$.

e) $f_t(x) = \frac{x+t}{(x-t)^2}$ $\Rightarrow$ Domain = $\mathbb{R}\backslash\{t\}$ with $S_y(0|\frac{1}{t})$, $S_x(-t|0)$ with sign change, vertical asymptote without sign change at x = t and horizontal asymptote y = 0. $f_t'(x) = -\frac{x+3t}{(x-t)^3}$ and $f_t''(x) = \frac{2x-8t}{(x-t)^4}$ $\Rightarrow$ Min(−3t|− $\frac{1}{8t}$) with locus $y = \frac{3}{8x}$ and Inf(4t| $\frac{5}{4t}$) with locus $y = \frac{5}{x}$.

f) $f_t(x) = x + 3t + \frac{4t^2}{x-t} = \frac{(x+t)^2}{x-t}$ ⇒ Domain D = $\mathbb{R}\backslash\{t\}$ with $S_y(0|-1)$, $S_x(-t|0)$ with sign change, vertical asymptote with sign change at x = t and asymptotic line y = x + 3t. $f_t'(x) = 1 - \frac{4t^2}{(x-t)^2} = \frac{x^2-2tx-3t^2}{(x-t)^2}$ and $f_t''(x) = \frac{8t^2}{(x-t)^3}$ ⇒ Max(−t|0) with locus y = 0 and Min(3t|8t) with locus $y = \frac{8}{3}x$.

Exercise 12: Optimization problems with rational functions

a) Circumference C = 2a + 2b with area A = a·b = 36 ⇒ $C(a) = 2a + \frac{72}{a}$ with $C'(a) = 2 - \frac{72}{a^2} = 0$ ⇔ a = ± 6, but Domain D_A =]0; ∞[. Since C(a) → ∞ for a → 0 and a → ∞ the absolute Min is at a = b = 6 cm with C(6) = 24 cm.

b) Circumference C = 2r + s with area $A = \frac{1}{2} \cdot s \cdot r = 100$ ⇒ $C(r) = 2r + \frac{200}{r}$ with $C'(r) = 2 - \frac{200}{r^2} = 0$ ⇔ r = ± 10, but Domain D_A =]0; ∞[. Since C(r) →∞ for r → 0 and r → ∞ the absolute Min is at r = 10 cm and s = 20 cm with C = 40 cm.

c) The area $A(x) = \sqrt{x^2 + \frac{4}{x^4}}$ is minimal if and only if $B(x) = x^2 + \frac{4}{x^4}$ is minimal. $B'(x) = 2x - \frac{16}{x^5} = 0$ yields $x = \pm\sqrt{2}$ with Domain D_A =]0; ∞[. Since B(x) → ∞ for x → 0 and x → ∞ the absolute Min is at $x = \sqrt{2}$ with $B(\sqrt{2}) = 3$ cm² resp. $A(\sqrt{2}) = \sqrt{3}$ cm.

d) Area $A(x) = 2x \cdot f(x) = \frac{40x}{x^2+5}$ with $A'(x) = \frac{200-40x^2}{(x^2+5)^2} = 0$ ⇒ $x = \pm\sqrt{5}$ with Domain D_A =]0; ∞[. Since A(x) → ∞ for x → 0 and x → ∞ the absolute Max is at $x = \sqrt{5}$ cm with $A(\sqrt{5}) = 4\sqrt{5}$ cm².

e) Circumference C = 2b + 2r + πr with Area $A = 2rb + \frac{\pi}{2}r^2 = 8$ ⇔ $b = \frac{4}{r} - \frac{\pi}{4}r$ ⇒ $C(r) = \frac{\pi}{2}r + 2r + \frac{8}{r}$ with $C'(r) = \frac{\pi}{2} + 2 - \frac{8}{r^2} = 0$ for $r = \pm\frac{4}{\sqrt{4+\pi}} \approx \pm 1{,}5$. Since $A = 2rb + \frac{\pi}{2}r^2 = 8$ we have $r \le \frac{4}{\sqrt{\pi}}$, so the Domain is $D_C =]0; \frac{4}{\sqrt{\pi}}] \approx]0; 2{,}26]$. Since C(r) → ∞ for r → 0^+ and $C(\frac{4}{\sqrt{\pi}}) = 4\sqrt{\pi} + \frac{8}{\sqrt{\pi}} \approx 11{,}6$ m the absolute Min is at $r = \frac{4}{\sqrt{4+\pi}} \approx 1{,}5$ m with $C(\frac{4}{\sqrt{4+\pi}}) = 4\sqrt{4+\pi} \approx 10{,}69$ m.

f) Surface area $S = 2\pi rh + 2\pi r^2$ with Volume $V = \pi r^2 h = 1000$ cm³ ⇔ $h = \frac{1000}{\pi r^2}$ ⇒ $S(r) = 2\pi r^2 + \frac{2000}{r}$ with $S'(r) = 4\pi r - \frac{2000}{r^2} = 0$ for $r = \sqrt[3]{\frac{500}{\pi}} \approx 5{,}42$ cm, with Domain D_S =]0; ∞[. Since S(r) → ∞ für r → 0 and r → ∞ the absolute Min is at $r = \sqrt[3]{\frac{500}{\pi}} \approx 5{,}42$ cm and h = 10,86 cm with $S(\sqrt[3]{\frac{500}{\pi}}) \approx 553{,}58$ cm². As comparison: A 1 Litre cube has a side length a = 10 cm and a surface area of $S_W = 6a^2 = 600$ cm²!

g) Surface area $S = S_{Hemisphere} + S_{Bottom} + S_{lateral\ surface} = 2\pi r^2 + \pi r^2 + 2\pi rh = 3\pi r^2 + 2\pi rh$ with Volume $V = \frac{2}{3}\pi r^3 + \pi r^2 h = 1000$ cm³ ⇒ $h = \frac{1000}{\pi r^2} - \frac{2r}{3}$ ⇒ $S(r) = \frac{5}{3}\pi r^2 + \frac{2000}{r}$ with $S'(r) = \frac{10}{3}\pi r - \frac{2000}{r^2} = 0$ for $r = \sqrt[3]{\frac{600}{\pi}}$ cm ≈ 5,76 cm with Domain $D_S =]0; \sqrt[3]{\frac{1500}{\pi}}] =]0; 10\sqrt[3]{\frac{1{,}5}{\pi}}] \approx$]0 cm; 7,81 cm[. Since S(r) → ∞ for r → 0^+ and $S(10\sqrt[3]{\frac{1{,}5}{\pi}}) = 300\sqrt[3]{2{,}25\pi}$ cm² ≈ 575,7 cm² the absolute Min is at $r = \sqrt[3]{\frac{600}{\pi}}$ cm ≈ 5,76 cm and h ≈ 26,3 cm with $S(\sqrt[3]{\frac{600}{\pi}}) = 519{,}2$ cm².

h) Surface area $S = 2a^2 + 4ah$ with Volume $V = a^2h = 1000$ cm³ ⇒ $h = \frac{1000}{a^2}$ ⇒ $S(a) = 2a^2 + \frac{4000}{a}$ with $S'(a) = 4a - \frac{4000}{a^2} = 0$ for a = 10 cm, with Domain D_S =]0; ∞[. Since S(a) → ∞ for a → 0^+ and a → ∞, the absolute Min of S is at a = 10 cm and h = 10 cm with O(10) = 600 cm².