5.4. Differentiation rules

5.4.1. The substitution method (chain rule)

Theorem: Substitution method or chain rule of differentiation
The derivative of f(x) = **g(**z(x)**)** is f'(x) = z'(x)·**g'(**z(x)**)**
z'(x) is called the **inner derivative**
g'(z(x)**)** is called the **outer derivative**
Proof: The continuity of z(x) implies $\lim\limits_{\Delta x\to 0} z(x+\Delta x) = z(x)$ resp. $\lim\limits_{\Delta x\to 0} \Delta z = 0$ with Δz := z(x + Δx) – z(x) and therefore

$$\begin{array}{ccl}
\dfrac{f(x+\Delta x)-f(x)}{\Delta x} & = & \dfrac{g(z(x+\Delta x))-g(z(x))}{\Delta x}\\
\Bigg\downarrow & = & \cdot\,\dfrac{g(z(x+\Delta x))-g(z(x))}{z(x+\Delta x)-z(x)}\cdot\dfrac{z(x+\Delta x)-z(x)}{\Delta x}\\
\Delta x\to 0 & = & \cdot\,\dfrac{g(z+\Delta z)-g(z)}{\Delta z}\cdot\dfrac{z(x+\Delta x)-z(x)}{\Delta x}\\
\Bigg\downarrow & & \qquad\Big\downarrow\Delta x\to 0\qquad\quad\Big\downarrow\Delta x\to 0\\
f'(x) & = & \qquad g'(z(x))\quad\cdot\quad z'(x)
\end{array}$$

Example 1:

Find the derivative of f(x) = $(x^2 + 2)^5$.

Solution:

$f(x) = \mathbf{g(}z(x)\mathbf{)} \Rightarrow f'(x) = z'(x)\cdot\mathbf{g'(}z(x)\mathbf{)}$

$f(x) = \mathbf{(}x^2 + 2\mathbf{)^5} \Rightarrow f'(x) = 2x\cdot\mathbf{5(}x^2 + 2\mathbf{)^4}$

$= 10x(x^2 + 2)^4$

Example 2:

Find the derivative of f(x) = $\sqrt{x^2+5}$

Solution:

$f(x) = \mathbf{g(}z(x)\mathbf{)} \Rightarrow f'(x) = z'(x)\cdot\mathbf{g'(}z(x)\mathbf{)}$

$f(x) = \mathbf{(}x^2 + 5\mathbf{)^{0,5}} \Rightarrow f'(x) = 2x\cdot\mathbf{0{,}5\cdot(}x^2 + 5\mathbf{)^{-0,5}}$

$= \dfrac{x}{\sqrt{x^2+5}}$

Example 3:

Find the derivative of f(x) = $\dfrac{1}{x^2+5}$.

Solution:

$f(x) = \mathbf{g(}z(x)\mathbf{)} \Rightarrow f'(x) = z'(x)\cdot\mathbf{g'(}z(x)\mathbf{)}$

$f(x) = \mathbf{(}x^2 + 2\mathbf{)^{-1}} \Rightarrow f'(x) = 2x\cdot\mathbf{(-1)\cdot(}x^2 + 2\mathbf{)^{-2}}$

$= -\dfrac{2x}{(x^2+5)^2}$

Example 4:

Find the derivative of f(x) = $\sin(x^2 + 5)$

Solution:

$f(x) = \mathbf{g(}z(x)\mathbf{)} \Rightarrow f'(x) = z'(x)\cdot\mathbf{g'(}z(x)\mathbf{)}$

$f(x) = \mathbf{\sin(}x^2 + 5\mathbf{)} \Rightarrow f'(x) = 2x\cdot\mathbf{\cos(}x^2 + 5\mathbf{)}$

Substitution method in differential notation
Diffential notattion is widely used in physics and the applied sciences and constitutes a shortened and very intuitive notation of differential calculus. For ξ → x we write ξ − x → dx and f(ξ) − f(x) → df so that $\dfrac{f(\xi)-f(x}{\xi-x} \to \dfrac{df}{dx} = f'(x)$ The substitution method then appears in the simple form: $\dfrac{df}{dx} = \dfrac{df}{dz}\cdot\dfrac{dz}{dx}$. It is especially useful in its reverse use in integration.
(see 5.1.)

Exercises on differentiation rules No 1 and 2

5.4.2. The natural exponential function (see also 4.7.1.)

Definition and theorem on the natural exponential function

The **natural exponential function** $\exp(x) = e^x$ with the **Euler number** $e = \lim\limits_{\Delta x \to 0} \sqrt[\Delta x]{1+\Delta x} = 2{,}71828....$ is the only function which is identical to its own derivative, i.e. ist y-value is equal to its gradient at every x. Its inverse is the **natural logarithm** $\ln(x) = \log_e(x)$.

Proof: If $(e^x)' = e^x$, **then** for $\Delta x \to 0$ follows

$$\lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \Leftrightarrow \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1 \Leftrightarrow e^{\Delta x} - 1 \to \Delta x \Leftrightarrow e^{\Delta x} \to 1 + \Delta x \Leftrightarrow \sqrt[\Delta x]{1+\Delta x} \to e \text{ for } \Delta x \to 0.$$

Derivative of the general exponential function

Any exponential function $y = a^x$ with base $a > 0$ and $a \neq 1$ can be expressed as a natural exponential function with inner factor: $a^x = (e^{\ln(a)})^x = e^{x\cdot\ln(a)}$.
Thus the **chain rule** can be used to get the derivative: $(a^x)' = \ln(a)\cdot e^{x\cdot\ln a} = \ln(a)\cdot e^x$

Example: $f(x) = \mathbf{g(}z(x)\mathbf{)} = \mathbf{e}^{x^2+5} \Rightarrow f'(x) = z'(x)\cdot\mathbf{g'(}z(x)\mathbf{)} = 2x\cdot \mathbf{e}^{x^2+5}$

Exercises on differentiation rules No. 3

5.4.3. The product rule

Sums and **constant factors** are not affected by the process of differentiation. Unfortunately **products** are a lot more sensitive:

Theorem: Product rule
The derivative of $f(x) = g(x)\cdot h(x)$ is $f'(x) = g'(x)\cdot h(x) + g(x)\cdot h'(x)$. **Short notation**: $(gh)' = g'h + gh'$
Proof

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{g(x+\Delta x)\cdot h(x+\Delta x) - g(x)\cdot h(x)}{\Delta x}$$

$$= \frac{g(x+\Delta x)\cdot h(x+\Delta x) - g(x)\cdot h(x+\Delta x) + g(x)\cdot h(x+\Delta x) - g(x)\cdot h(x)}{\Delta x}$$

$$= \frac{[g(x+\Delta x) - g(x)]\cdot h(x+\Delta x) + g(x)\cdot[h(x+\Delta x) - h(x_0)]}{\Delta x}$$

$$\Big\downarrow \Delta x \to 0 \qquad = \frac{g(x+\Delta x) - g(x)}{\Delta x}\cdot h(x+\Delta x) + g(x)\cdot \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

$$\qquad\qquad \Big\downarrow \Delta x \to 0 \qquad\qquad\qquad \Big\downarrow \Delta x \to 0$$

$$f'(x) = g'(x)\cdot h(x) + g(x)\cdot h'(x)$$

Example 1:
Find the derivative of $f(x) = (x^2 + 2)\cdot\sin x$
Solution:

$f(x) = \mathbf{g(x)}\cdot h(x) \Rightarrow f'(x) = \mathbf{g'(x)\cdot}h(x) + \mathbf{g(x)\cdot}h'(x)$

$f(x) = \mathbf{(x^2 + 2)}\cdot\sin x \Rightarrow f'(x) = \mathbf{2x}\cdot\sin x + \mathbf{(x^2 + 2)}\cdot\cos x$

Example 2:
Find the derivative of $f(x) = (x^2 + 5)\cdot e^x$
Solution:

$f(x) = \mathbf{g(x)}\cdot h(x) \Rightarrow f'(x) = \mathbf{g'(x)\cdot}h(x) + \mathbf{g(x)\cdot}h'(x)$

$f(x) = \mathbf{(x^2 + 2)}\cdot e^x \Rightarrow f'(x) = \mathbf{2x}\cdot e^x + \mathbf{(x^2 + 2)}\cdot e^x$
$= (x^2 + 2x + 2)\cdot e^x$

Example 3:
Find the derivative of $f(x) = (2x + 2)\cdot\sqrt{x}$
Solution:

$f(x) = \mathbf{g(x)}\cdot h(x) \Rightarrow f'(x) = \mathbf{g'(x)\cdot}h(x) + \mathbf{g(x)\cdot}h'(x)$

$f(x) = \mathbf{(x^2 + 2)}\cdot\sqrt{x} \Rightarrow f'(x) = \mathbf{2x}\sqrt{x} + \mathbf{(x^2+2)}(-\frac{1}{2\sqrt{x}})$

$= \frac{3}{2}x\sqrt{x} - \frac{1}{\sqrt{x}}$

Example 4:
Find the derivative of $f(x) = (x + 1)\cdot e^{x^2+5}$
Solution:

$f(x) = \mathbf{g(x)}\cdot h(x) \Rightarrow f'(x) = \mathbf{g'(x)\cdot}h(x) + \mathbf{g(x)\cdot}h'(x)$

$f(x) = \mathbf{(x + 1)}\cdot e^{x^2+5} \Rightarrow f'(x) = \mathbf{1}\,e^{x^2+5} + \mathbf{(x + 1)}\cdot 2x\,e^{x^2+5}$

$= (2x^2 + 2x + 1)\cdot e^{x^2+5}$

Exercises on differentiation rules No. 4 and 5

5.4.4. The natural logarithm function (see also 4.7.2.)

Theorem on the differentiation of inverse function

The function y = f(x) with derivative f'(x) and inverse x = $f^{-1}(y)$ has the derivative $[f^{-1}(y)]' = \frac{1}{f'(f^{-1}(y))}$.

Proof: A for the proof of the chain rule in 5.4.1 we use
the continuity of f(x) in the form $\lim\limits_{\Delta x \to 0} f(x + \Delta x) = f(x)$ resp. $\lim\limits_{\Delta x \to 0} \Delta y = 0$ with $\Delta y := f(x + \Delta x) - f(x)$ and
the continuity of $f^{-1}(y)$ in the form $\lim\limits_{\Delta y \to 0} f^{-1}(y + \Delta y) = f^{-1}(y)$ resp. $\lim\limits_{\Delta y \to 0} \Delta x = 0$ with $\Delta x := f^{-1}(y + \Delta y) - f^{-1}(y)$

which result in

$$[f^{-1}(y)]' = \lim_{\Delta y \to 0} \frac{f^{-1}(y + \Delta y) - f^{-1}(y)}{\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{f(x + \Delta x) - f(x)} = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

Derivative of the natural logarithm function

with $f(x) = f'(x) = e^x$ and $f^{-1}(y) = \ln(y)$ we obtain by the above theorem $\ln'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f(f^{-1}(y))} = \frac{1}{y}$.

Derivative of the general logarithm function

The change of base $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ is done by multiplication with the factor $\frac{1}{\ln(a)}$ which remains unaffected by differentiation: $\log_a'(x) = \frac{1}{x \cdot \ln(a)}$.

Example: f(x) = **g(**z(x)**)** = **ln(**$x^2 + 5$**)** ⇒ $f'(x) = z'(x) \cdot$ **g'(**z(x)**)** $= 2x \cdot \frac{1}{x^2 + 5}$

Exercises on differentiation rules No. 3

5.4.5. The quotient rule

Theorem: Quotient rule

The function $f(x) = \frac{g(x)}{h(x)}$ with $h(x) \neq 0$ has the derivative $f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2}$ **short notation:**: $(\frac{g}{h})' = \frac{g'h - gh'}{h^2}$

Proof

We write the quotient as aproduct $f(x) = g(x) \cdot \frac{1}{h(x)}$ and apply the **chain rule** together with the **product rule:**

$$f'(x) = g'(x) \cdot \frac{1}{h(x)} + g(x) \cdot \left(\frac{1}{h(x)}\right)' = g'(x) \cdot \frac{1}{h(x)} + g(x) \cdot \left(-\frac{h(x)}{(h(x))^2}\right) = \frac{g'(x)}{h(x)} - \frac{g(x) \cdot h'(x)}{(h(x))^2} = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2}$$

Example:

Determine the derivative of $f(x) = \frac{x^2 + 5}{3x - 1}$

Solution:

$$f(x) = \frac{\mathbf{g(x)}}{h(x)} \quad \Rightarrow f'(x) = \frac{\mathbf{g'(x)} \cdot h(x) - \mathbf{g(x)} \cdot h'(x)}{(h(x))^2}$$

$$f(x) = \frac{\mathbf{x^2 + 5}}{3x - 1} \quad \Rightarrow f'(x) = \frac{\mathbf{2x} \cdot (3x - 1) - \mathbf{(x^2 + 5)} \cdot 3}{(3x - 1)^2} = \frac{3x^2 - 17x - 15}{9x^2 - 6x + 1}$$

Exercises on differentiation rules No. 7 - 11
Exercises on the curve analysis of compound functions No. 1 - 9