5.5. Exercises on integration

Exercise 1: Antiderivatives

Give all possible antiderivatives o f:

a) $f(x) = 0$

b) $f(x) = 1$

c) $f(x) = 2$

d) $f(x) = a \in \mathbb{R}$

e) $f(x) = x$

f) $f(x) = x^2$

g) $f(x) = x^3$

h) $f(x) = x^{-3}$

i) $f(x) = x^{-2}$

j) $f(x) = x^{-1}$

k) $f(x) = x^n$ with $n \in \mathbb{R}\backslash\{-1\}$

l) $f(x) = 5x^2 - 3x + 6$

m) $f(x) = x^4 - x^3 + x^2 - x + 1$

n) $f(u) = 4u^3 - 3u^2 + 7u$

o) $f(x) = \frac{3}{2}x^2 - 3x + \sqrt{x} - 5$

p) $f(x) = 16x^4 + x - 7 + \frac{5}{x^2} - \frac{30}{x^3}$

q) $f(t) = \frac{3}{2}t - \frac{1}{2\sqrt{t}}$

r) $f(x) = a_n x^n + a_{n-1}x^{x-1} + ... + a_1 x + a_0$

s) $f(t) = \sin(t)$

t) $f(t) = \cos(t)$

Exercise 2: Fundamental theorem and properties of the integral

Determine the following integrals:

a) $\int_{-1}^{1}(-\frac{1}{2}x^2 - x + \frac{3}{2})dx$

b) $\int_{-1}^{2}(x^3 + x^2)dx$

c) $\int_{-3}^{-2}(x^2 + 3x + 2)dx$

d) $\int_{1}^{2}x^2dx$, $\int_{2}^{3}x^2dx$ und $\int_{1}^{3}x^2dx$ (additivity)

e) $\int_{2}^{1}x^2dx$ (Exchange of limits resp.. dx < 0)

f) $\int_{0}^{3}(x^2 - 4x + 3)dx$ (areas below the x-axis. f(x) < 0)

Exercise 3: Areas below the x-axis

Find the absolute measure of the area bounded by x = a, x = b, the x-axis and the curve y = f(x).

a) $f(x) = x^2 - 1$ with $a = -1$ and $b = 2$

b) $f(x) = -x^2 - 4x - 3$ with $a = -4$ and $b = -1$

c) $f(x) = x^3$ with $a = -1$ and $b = 2$

d) $f(x) = x^3 - x$ with $a = -1$ and $b = 1$

e) $f(x) = x^3 - x$ with $a = -1$ and $b = 2$

f) $f(t) = \sin t$ with $a = -\pi$ and $b = \pi$

Exercise 4: Areas below the x-axis

Find the absolute measure of the areas between the curves y = f(x) and the x-axis.

a) $f(x) = -x^2 + 1$ b) $f(x) = x^3 - 4x$ c) $f(x) = -x^4 + x^2$ d) $f(x) = x^3 - 3x^2 + 2x$

Exercise 5: Areas between two curves

Find the absolute measure of the areas bounded by the curves y = f(x) and y = g(x) resp. the vertical lines x = a and x = b:

a) $f(x) = -x^2 + 2$, $g(x) = -x^2 + 3$, $a = -1$ und $b = 1$

b) $f(x) = x^2$, $g(x) = 2$, $a = -1$ und $b = 1$

c) $f(x) = 3x$, $g(x) = -x + 2$, $a = 0$ und $b = 1$

d) $f(x) = x^2$, $g(x) = x$, $a = 0$ und $b = 2$

e) $f(x) = x^2$, $g(x) = x^3$, $a = -2$ und $b = -1$

f) $f(t) = \sin t$, $g(t) = \cos t$, $a = -\frac{\pi}{4}$, $b = \frac{\pi}{4}$

Exercise 6: Areas between two curves

Find the absolute measure of the areas between the curves y = f(x) and y = g(x).

a) $f(x) = x^2$, $g(x) = 2 - x^2$ b) $f(x) = x^3$, $g(x) = x^2$ c) $f(x) = x^3$, $g(x) = x$ d) $f(x) = x^3 - 3x$, $g(x) = 2x^2$

Exercise 7: Variable limits

Find t so that A(t) = 2.

a) $A(t) = \int_{0}^{t}(-x^2 + \frac{7}{3})dx$

b) $A(t) = \int_{0}^{t}(x^2 + \frac{4}{3}x + 1)dx$

c) $A(t) = \int_{1}^{t}(x^2 - \frac{1}{3})dx$

d) $A(t) = \int_{2}^{t}(x^2 - \frac{4}{3}x - 1)dx$

e) $A(t) = \int_{1}^{2}(x^2 + t)dx$

f) $A(t) = \int_{0}^{t}(x^2 - \frac{2}{3}tx + 2)dx$

Exercise 8: Integration by substitution

Find the value of the following integrals by substituting $dz = \frac{dz}{dx}dx$ with a suitable z(x):

a) $\int_0^3 (2x \cdot e^{x^2-2})dx$

b) $\int_1^2 (4x \cdot e^{x^2-4})dx$

c) $\int_0^1 e^{3x+1}dx$

d) $\int_0^5 e^{-x}dx$

e) $\int_{-3}^0 e^{1-x}dx$

f) $\int_0^9 \frac{1}{(x+1)^2}dx$

g) $\int_0^1 \frac{4x-10}{(x^2-5x+6)^2}dx$

h) $\int_0^{0,5} \frac{2x}{x^4-2x^2+1}dx$

i) $\int_0^1 \frac{1}{x+1}dx$

j) $\int_2^4 \frac{x}{x^2-1}dx$

k) $\int_0^1 \left(x - \frac{2tx}{x^2+t}\right)dx$

l) $\int_2^3 \frac{11x-4}{x-1}dx$

m) $\int_0^1 \frac{x^2+t}{x+t}dx$

n) $\int_{-1}^1 \frac{t^2-1}{t^2}x dx$

o) $\int_1^2 \ln x \; dx$

p) $\int_0^1 x \cdot \ln(x^2+1) \; dx$

q) $\int_2^3 \frac{(\ln x)^2}{x}dx$

r) $\int_1^2 \frac{\sqrt{\ln x}}{x}dx$

s) $\int_e^{2e} \frac{\ln x}{x}dx$

t) $\int_0^1 \frac{\ln(x+1)}{x+1}dx$

Exercise 9: Integration by parts

Find the value of the following integrals by using the product rule:

a) $\int_0^2 (x \cdot e^x)dx$

b) $\int_0^1 (x^2 \cdot e^x)dx$

c) $\int_{-1}^1 (x \cdot e^{2x})dx$

d) $\int_1^2 (x \cdot e^{-x})dx$

e) $\int_0^1 (x^2 \cdot e^{-x})dx$

f) $\int_e^{e^2} x \cdot \ln x \; dx$

g) $\int_1^3 (\ln x)^2 \; dx$

h) $\int_1^2 \frac{\ln x}{x}$

Exercise 10: Integration by substitution and by parts

Show by integration that F(x) is an antiderivative of f(x).

a) $f(x) = 4e^{2x}$ with $F(x) = 2e^{2x}$

b) $f(x) = e^{-0,5x-1}$ with $F(x) = -2e^{-0,5x-1}$

c) $f(x) = -6e^{-3x+1}$ with $F(x) = 2e^{-3x+1}$

d) $f(x) = 2x \cdot e^{0,5x^2}$ with $F(x) = 2\,e^{0,5x^2}$

e) $f(x) = -6x \cdot e^{x^2+1}$ with $F(x) = -3\,e^{x^2+1}$

f) $f(x) = (4x-2)\,e^{-x^2+x-1}$ with $F(x) = -2\,e^{-x^2+x-1}$

g) $f(x) = (x+3)e^x$ with $F(x) = (x+2)e^x$

h) $f(x) = (-2x-1)e^x$ with $F(x) = (-2x+1)e^x$

i) $f(x) = x^2e^x$ with $F(x) = (x^2-2x+2)e^x$

j) $f(x) = (x+3)e^{-x}$ with $F(x) = -(x+4)e^{-x}$

k) $f(x) = (-3x+11)e^{3x}$ with $F(x) = (-x+4)e^{3x}$

l) $f(x) = -(6x+1)e^{-3x+1}$ with $F(x) = (2x+1)e^{-3x+1}$

m) $f(x) = (2x-5)e^{2x+1}$ with $F(x) = (x-3)e^{2x+1}$

n) $f(x) = \frac{x}{x^2+1}$ with $F(x) = \frac{1}{2}\ln(x^2+1)$

o) $f(x) = \frac{1}{(2x+4)^2}$ with $F(x) = -\frac{1}{4x+8}$

p) $f(x) = 1 + \ln(x)$ with $F(x) = x \cdot \ln(x)$

q) $f(x) = (\ln(x))^2$ with $F(x) = x \cdot (\ln(x))^2 - 2x \cdot \ln(x) + 2x$

r) $f(x) = 1 - (\ln(1-x))^2$ with $F(x) = (1-x) \cdot (1-\ln(1-x))^2$

Exercise 11: Improper integrals

Write the following improper integrals as limits and find their values:

a) $\int_1^\infty \frac{1}{x^2}dx$

b) $\int_0^\infty \frac{1}{(x+1)^2}dx$

c) $\int_1^\infty \frac{1}{x^{1,5}}dx$

d) $\int_0^1 \frac{1}{x^{0,5}}dx$

e) $\int_{-1}^0 \frac{1}{\sqrt{x+1}}dx$

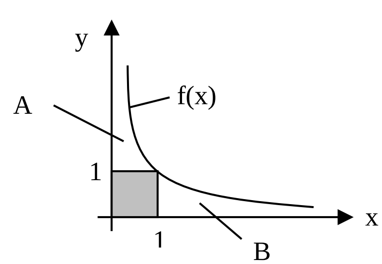
f) $\int_0^\infty e^{-x}dx$

g) $\int_0^\infty x \cdot e^{-x^2}dx$

h) $\int_0^1 \frac{1}{x^2} \cdot e^{-\frac{1}{x}}dx$

Exercise 12: Improper integrals

Find all possible n > 0 so that the areas A resp. B between the hyperbola $f(x) = x^{-n}$ and the y-axis resp. the x-axis have finite measure and give an expression for this measure depending on t.



5.5. Solutions to the exercises on integration

Exercise 1: Antiderivatives ($c \in \mathbb{R}$)

a) $F_C(x) = c$

b) $F_C(x) = x + c$

c) $F_C(x) = 2x + c$

d) $F_C(x) = ax + c$

e) $F_C(x) = \frac{1}{2}x^2 + c$

f) $F_C(x) = \frac{1}{3}x^3 + c$

g) $F_C(x) = \frac{1}{4}x^4 + c$

h) $F_C(x) = -\frac{1}{2}x^{-2} + c$

i) $F_C(x) = -x^{-1} + c$

j) $F_C(x) = \ln x + c$

k) $F_C(x) = \frac{1}{n+1}x^{n+1} + c$

l) $F_C(x) = \frac{5}{3}x^3 - \frac{3}{2}x^2 + 6x + c$

m) $F_C(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x + c$

n) $F_C(u) = u^4 - u^3 + \frac{7}{2}u + c$

o) $F_C(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{2}{3}x^{1,5} - 5x + c$

p) $F_C(x) = \frac{16}{5}x^5 + \frac{1}{2}x^2 - 7x + c - \frac{5}{x} + \frac{15}{x^2}$

q) $F_C(t) = \frac{3}{4}t^2 - \sqrt{t} + c$

r) $F_C(x) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + ... + \frac{a_1}{2}x^2 + a_0x + c$

s) $F_c(t) = -\cos t + c$

t) $F_c(t) = \sin t + c$

Exercise 2: Fundamental theorem of calculus and properties of the integral

a) $\int_{-1}^{1}(-\frac{1}{2}x^2 - x + \frac{3}{2})dx = \left[-\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{3}{2}x\right]_{-1}^{1} = -\frac{1}{6}\left[x^3 + 3x^2 - 9x\right]_{-1}^{1} = -(-\frac{5}{6}) - (-\frac{11}{6}) = \frac{8}{3}$

b) $\int_{-1}^{2}(x^3 + x^2)dx = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3\right]_{-1}^{2} = (4 + \frac{8}{3}) - (\frac{1}{4} - \frac{1}{3}) = 6\frac{3}{4}$

c) $\int_{-3}^{-2}(x^2 + 3x + 2)dx = \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right]_{-3}^{-2} = \frac{1}{6}\left[2x^3 + 9x^2 + 12x\right]_{-3}^{-2} = \frac{5}{6}$

d) $\int_{1}^{2}x^2dx = \frac{7}{3}$, $\int_{2}^{3}x^2dx = \frac{19}{3}$ and $\int_{1}^{3}x^2dx = \frac{26}{3}$

e) $\int_{2}^{1}x^2dx = -\frac{7}{3}$

f) $A = \left|\int_{0}^{1}(x^2 - 4x + 3)dx\right| + \left|\int_{1}^{3}(x^2 - 4x + 3)dx\right| = |\frac{4}{3}| + |-\frac{4}{3}| = \frac{8}{3}$

Exercise 3: Areas below the x-axis

a) $A = \left|\int_{-1}^{1}(x^2 - 1)dx\right| + \left|\int_{1}^{2}(x^2 - 1)dx\right| = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

b) $A = \left|\int_{-4}^{-3}(-x^2 - 4x - 3)dx\right| + \left|\int_{-3}^{-1}(-x^2 - 4x - 3)dx\right| = |-\frac{4}{3}| + |\frac{4}{3}| = \frac{8}{3}$

c) $A = \left|\int_{-1}^{0}x^3dx\right| + \left|\int_{0}^{2}x^3dx\right| = \frac{1}{4} + 4 = \frac{17}{4}$

d) $A = \left|\int_{-1}^{0}(x^3 - x)dx\right| + \left|\int_{0}^{1}(x^3 - x)dx\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

e) $A = \left|\int_{-1}^{0}(x^3 - x)dx\right| + \left|\int_{0}^{1}(x^3 - x)dx\right| + \left|\int_{1}^{2}(x^3 - x)dx\right| = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{3}$

f) $A = 2\left|\int_{0}^{\pi}(\sin t)dt\right| = 4$

Exercise 4: Areas below the x-axis

a) $A = 2\cdot\int_{0}^{1}(-x^2 + 1)dx = 2\cdot\left[-\frac{1}{3}x^3 + x\right]_0^1 = 2\cdot\frac{2}{3} = \frac{4}{3}$

b) $A = 2\cdot\left|\int_{0}^{2}(x^3 - 4x)dx\right| = 2\cdot\left|\left[\frac{1}{4}x^4 - 2x^2\right]_0^2\right| = 2\cdot|-4| = 8$

c) $A = 2\cdot\int_{0}^{1}(-x^4 + x^2)dx = 2\cdot\left[-\frac{1}{5}x^5 + \frac{1}{3}x^2\right]_0^1 = 2\cdot\frac{2}{15} = \frac{4}{15}$

d) $A = \int_{0}^{1}(x^3 - 3x^2 + 2x)dx + \left|\int_{1}^{2}(x^3 - 3x^2 + 2x)dx\right| = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^1 + \left|\left[\frac{1}{4}x^4 - x^3 + x^2\right]_1^2\right| = \frac{1}{4} + \left|-\frac{1}{4}\right| = \frac{1}{2}$

Exercise 5: Areas between two curves

a) $A = \int_{-1}^{1}1dx = 2$

b) $A = \left|\int_{-1}^{1}(x^2 - 2)dx\right| = 3\frac{1}{3}$

c) $A = \left|\int_{0}^{0,5}(4x - 2)dx\right| + \left|\int_{0,5}^{1}(4x-2)dx\right| = \frac{1}{2} + \frac{1}{2} = 1$

d) $A = \left|\int_{0}^{1}(x^2 - x)dx\right| + \left|\int_{1}^{2}(x^2 - x)dx\right| = \frac{1}{6} + \frac{5}{6} = 1$

e) $A = \left|\int_{-2}^{-1}(x^3 - x^2)dx\right| = 6\frac{1}{12}$

f) $A = \left|\int_{-0,75\pi}^{0,25\pi}(\cos t - \sin t)dt\right| = 2\sqrt{2}$

Exercise 6: Areas between two curves

a) $A = \left|\int_{-1}^{1}(2x^2 - 2)dx\right| = \frac{8}{3}$

b) $A = \left|\int_{0}^{1}(x^3 - x^2)dx\right| = \frac{1}{12}$

c) $A = \left|\int_{-1}^{0}(x^3 - x)dx\right| + \left|\int_{0}^{1}(x^3 - x)dx\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

d) $A = \left|\int_{-1}^{0}(x^3 - 2x^2 - 3x)dx\right| + \left|\int_{0}^{3}(x^3 - 2x^2 - 3x)dx\right| = \frac{7}{12} + \frac{45}{4} = \frac{71}{6}$

Exercise 7: Variable limits

a) $A(t) = \int_0^t (-x^2 + \frac{7}{3})dx = -\frac{1}{3}t^3 + \frac{7}{3}t = 2 \Leftrightarrow t^3 - 7t + 6 = 0 \Leftrightarrow t = 1$

b) $A(t) = \int_0^t (x^2 + \frac{4}{3}x + 1)dx = \frac{1}{3}t^3 + \frac{2}{3}t^2 + t = 2 \Leftrightarrow t^3 - 2t^2 + 3t - 6 = 0 \Rightarrow t = 2$

c) $A(t) = \int_1^t (x^2 - \frac{1}{3})dx = \frac{1}{3}t^3 - \frac{1}{3}t - \frac{1}{3} + \frac{1}{3} = 2 \Leftrightarrow t^3 - t - 6 = 0 \Rightarrow t = 2$

d) $A(t) = \int_2^t (x^2 - \frac{4}{3}x - 1)dx = \frac{1}{3}t^3 - \frac{2}{3}t^2 - t - \frac{8}{3} + \frac{8}{3} + 2 = 2 \Leftrightarrow t^3 - 2t^2 - 3t = 0 \Rightarrow t = 3$

e) $A(t) = \int_1^2 (x^2 + t)dx = \frac{1}{3}2^3 + \frac{t}{2}2^2 - \frac{1}{3}1^3 - \frac{t}{2}1^2 = \frac{7}{3} + \frac{3}{2}t = 2 \Leftrightarrow 14 + 9t = 12 \Rightarrow t = -\frac{2}{9}$

f) $A(t) = \int_0^t (x^2 - \frac{2}{3}tx + 2)dx = \frac{1}{3}t^3 - \frac{t}{3}t^2 + 2t = 2 \Leftrightarrow 2t = 2 \Rightarrow t = 1$

Exercise 8: Integration by substitution

a) $\int_0^3 (2x \cdot e^{x^2-2})dx = \int_{-2}^7 e^z dz = e^7 - e^{-2} \approx 1096{,}50$ with $z(x) = x^2 - 2$

b) $\int_1^2 (4x \cdot e^{x^2-4})dx = 2\int_1^2 2x \cdot e^{x^2-4}dx = 2\int_{-3}^0 e^z dz = 2(1 - e^{-3}) \approx 1{,}90$ with $z(x) = x^2 - 4$

c) $\int_0^1 e^{3x+1}dx = \frac{1}{3}\int_0^1 3 \cdot e^{3x+1}dx = \frac{1}{3}\int_1^4 e^z dz = \frac{1}{3}(e^4 - e) \approx 17{,}29$ with $z(x) = 3x + 1$

d) $\int_0^5 e^{-x}dx = -\int_0^5 (-1) \cdot e^{-x}dx = -\int_0^{-5} e^z dz = -(e^{-5} - 1) \approx 0{,}99$ with $z(x) = -x$

e) $\int_{-3}^0 e^{1-x}dx = -\int_{-3}^0 (-1) \cdot e^{1-x}dx = -\int_4^1 e^z dz = -(e - e^4) \approx 51{,}88$ with $z(x) = 1 - x$

f) $\int_0^9 \frac{1}{(x+1)^2}dx = \int_1^{10} \frac{1}{z^2}dz = \left[-\frac{1}{z}\right]_1^{10} = 0{,}9$ with $z(x) = x + 1$

g) $\int_0^1 \frac{4x - 10}{(x^2 - 5x + 6)^2}dx = \left[-\frac{2}{x^2 - 5x + 6}\right]_0^1 = -\frac{2}{3}$

h) $\int_0^{0,5} \frac{2x}{x^4 - 2x^2 + 1}dx = \int_0^{0,5} \frac{2x}{(x^2 - 1)^2}dx = \left[-\frac{1}{x^2 - 1}\right]_0^{0,5} = \frac{1}{3}$

i) $\int_0^1 \frac{1}{x+1}dx = \ln(2)$

j) $\int_2^4 \frac{x}{x^2 - 1}dx = \left[\frac{1}{2}\ln(x^2 - 1)\right]_2^4 = \frac{1}{2}\ln(5)$

k) $\int_0^1 \left(x - \frac{2tx}{x^2 + t}\right)dx = \int_0^1 x dx - \int_t^{1+t} \frac{t}{z}dz = \left[\frac{1}{2}x^2\right]_0^1 - t\ln(z)\Big|_t^{1+t} = \frac{1}{2} - t(\ln(1 + t) - \ln(t))$ with $z(x) = x^2 + t$

l) $\int_2^3 \frac{11x - 4}{x - 1}dx = \int_1^2 \frac{11z + 7}{z}dz = \int_1^2 \left(11 + \frac{7}{z}\right)dz = 11z + 7\ln(z)\Big|_1^2 = 11 + 7\ln(2) \approx 15{,}85$ with $z(x) = x - 1$

m) $\int_0^1 \frac{x^2 + t}{x + t}dx = \int_t^{1+t} \frac{z^2 - 2tz + t^2 + t}{z}dz = \int_t^{1+t} \left(z - 2t + \frac{t^2 + t}{z}\right)dz = \left[\frac{1}{2}z^2 - 2tz + (t^2 + t)\ln(z)\right]_t^{1+t} = \frac{1}{2} - t + (t^2 + t)\ln\left(1 - \frac{1}{t}\right)$

n) $\int_{-1}^{1} \frac{t^2-1}{t^2} x \; dx = \frac{t^2-1}{t^2} \int_{-1}^{1} x dx = \frac{t^2-1}{t^2} \left[\frac{1}{2} x^2 \right]_{-1}^{1} = 0$

o) $\int_{1}^{2} \ln(x) dx = x \cdot \ln(x) - x \,\Big|_1^2 = (2 \cdot \ln(2) - 2) - (0 - 1) \approx 0{,}39$ with antiderivative taken from formula booklet!

p) $\int_{0}^{1} x \cdot \ln(x^2+1) \; dx = \frac{1}{2} \int_{0}^{1} 2x \cdot \ln(x^2+1) \; dx = \int_{1}^{2} \ln(z) dz = \frac{1}{2} \ln(2) - \frac{1}{2} \approx 0{,}19$ with $z(x) = x^2 + 1$

q) $\int_{2}^{3} \frac{(\ln x)^2}{x} dx = \int_{\ln 2}^{\ln 3} z^2 dz = \frac{1}{3} (\ln(3))^3 - \frac{1}{3} (\ln(2))^3 \approx 0{,}33$ with $z(x) = \ln(x)$

r) $\int_{1}^{2} \frac{\sqrt{\ln x}}{x} dx = \int_{\ln 1}^{\ln 2} \sqrt{z} \; dz = \left[\frac{2}{3} z^{1,5} \right]_{0}^{\ln 2} = \frac{2}{3} (\ln(2))^{1,5} \approx 0{,}38$ with $z(x) = \ln(x)$

s) $\int_{e}^{2e} \frac{\ln x}{x} dx = \int_{\ln e}^{\ln 2e} z dz = \frac{1}{2} (\ln(2e))^2 - \frac{1}{2} \approx 0{,}93$ with $z(x) = \ln(x)$

t) $\int_{0}^{1} \frac{\ln(x+1)}{x+1} dx = \int_{\ln(1)}^{\ln(2)} z \cdot dz = \frac{1}{2} (\ln(2))^2 \approx 0{,}24$ with $z(x) = \ln(x + 1)$

Exercise 9: Integration by parts

a) $\int_{0}^{2} (x \cdot e^x) dx = \left[x \cdot e^x \right]_0^2 - \int_{0}^{2} (1 \cdot e^x) dx = \left[x \cdot e^x \right]_0^2 - \left[e^x \right]_0^2 = \left[(x-1) \cdot e^x \right]_0^2 = e^2 + 1 \approx 8{,}39$

b) $\int_{0}^{1} (x^2 \cdot e^x) dx = \left[x^2 \cdot e^x \right]_0^1 - \int_{0}^{1} (2x \cdot e^x) dx = \left[x^2 \cdot e^x \right]_0^1 - 2 \left[(x-1) \cdot e^x \right]_0^1 = \left[(x^2 - 2x + 2) \cdot e^x \right]_0^1 = e - 2 \approx 0{,}72$

(use result from a) !)

c) $\int_{-1}^{1} (x \cdot e^{2x}) dx = \left[x \cdot \frac{1}{2} e^{2x} \right]_{-1}^{1} - \int_{-1}^{1} (1 \cdot \frac{1}{2} e^{2x}) dx = \left[x \cdot \frac{1}{2} e^{2x} \right]_{-1}^{1} - \left[\frac{1}{4} e^{2x} \right]_{-1}^{1} = \left[(\frac{1}{2} x - \frac{1}{4}) \cdot e^x \right]_{-1}^{1} = \frac{1}{4} e^2 + \frac{3}{4} e^{-2} \approx 1{,}90$

d) $\int_{1}^{2} (x \cdot e^{-x}) dx = \left[x \cdot (-e^{-x}) \right]_1^2 - \int_{1}^{2} (1 \cdot (-e^{-x})) dx = \left[-x \cdot e^{-x} \right]_1^2 - \left[e^{-x} \right]_1^2 = \left[(-x-1) \cdot e^{-x} \right]_1^2 = -3e^{-2} + 2e^{-1} \approx 0{,}33$

e) $\int_{0}^{1} (x^2 \cdot e^{-x}) dx = \left[x^2 \cdot (-e^{-x}) \right]_0^1 - \int_{0}^{1} (2x \cdot (-e^{-x})) dx = \left[-x^2 \cdot e^{-x} \right]_0^1 + 2 \int_{0}^{1} (x \cdot e^{-x}) dx = \left[-x^2 \cdot e^{-x} \right]_0^1 + 2$

$\left[(-x-1) \cdot e^{-x} \right]_0^1 = \left[(-x^2 - 2x - 2) \cdot e^{-x} \right]_0^1 = -5 \cdot e^{-1} + 2 \approx 0{,}16$ (use result from d)!)

f) $\int_{e}^{e^2} x \cdot \ln(x) dx = \left[\frac{1}{2} x^2 \cdot \ln(x) \right]_e^{e^2} - \int_{e}^{e^2} \frac{1}{2} x^2 \cdot \frac{1}{x} \; dx = \left[\frac{1}{2} x^2 \cdot \ln(x) \right]_e^{e^2} - \left[\frac{1}{4} x^2 \right]_e^{e^2} = \left[\frac{1}{4} x^2 (2 \ln(x) - 1) \right]_e^{e^2} = \frac{1}{4} e^2(3e^2 - 1) \approx 39{,}10$

g) $\int_{1}^{3} (\ln(x))^2 \; dx = (x \ln(x) - x) \cdot \ln(x) \,\Big|_1^3 - \int_{1}^{3} (x \cdot \ln(x) - x) \cdot \frac{1}{x} \; dx = \left[x(\ln(x))^2 - x \ln(x) \right]_1^3 - \int_{1}^{3} (\ln(x) - 1) \; dx =$

$\left[x(\ln(x))^2 - x \ln x \right]_1^3 - (x \ln(x) - x - x \,\Big|_1^3 = \left[x(\ln(x))^2 - 2x \ln(x) + 2x \right]_1^3 = 3 \cdot (\ln(3))^2 - 6 \cdot \ln(3) + 6 - 2 \approx 1{,}03$

h) $\int_{1}^{2} \frac{\ln(x)}{x} dx = \int_{1}^{2} \frac{1}{x} \cdot \ln(x) dx = l(x) \cdot \ln(x) \,\Big|_1^2 - \int_{1}^{2} \ln(x) \cdot \frac{1}{x} dx \Rightarrow \int_{1}^{2} \frac{\ln(x)}{x} dx = \frac{1}{2} \; \ln(x) \cdot \ln(x) \,\Big|_1^2 = \frac{(\ln(2))^2}{2} \approx 0{,}24.$

Exercise 10: Finding antiderivatives using integration by substitution and by parts

a) $\int_a^b f(x)dx = \int_a^b 2\cdot 2e^{2x}dx = \int_{2a}^{2b} 2e^z dz = \left[2e^z\right]_{2a}^{2b} = \left[2\cdot e^{2x}\right]_a^b = F(x)\,_a^b$

b) $\int_a^b f(x)dx = \int_a^b (\frac{1}{-0,5})\cdot(-0,5)e^{-0,5x-1}dx = \int_{-0,5a-1}^{-0,5b-1} -2e^z dz = \left[-2e^z\right]_{-0,5a-1}^{-0,5b-1} = \left[-2e^{-0,5x-1}\right]_a^b = F(x)\,_a^b$

c) $\int_a^b f(x)dx = \int_a^b (-6)\cdot e^{-3x+1}dx = \int_a^b 2\cdot(-3)e^{-3x+1}dx = \int_{-3a+1}^{-3b+1} 2e^z dz = \left[2e^z\right]_{-3a+1}^{-3b+1} = \left[2\cdot e^{-3x+1}\right]_a^b = F(x)\,_a^b$

d) $\int_a^b f(x)dx = \int_a^b 2\cdot xe^{0,5x^2}dx = \int_{0,5a^2}^{0,5b^2} 2e^z dz = \left[2\cdot e^z\right]_{0,5a^2}^{0,5b^2} = \left[2\cdot e^{0,5x^2}\right]_a^b = F(x)\,_a^b$

e) $\int_a^b f(x)dx = \int_a^b (-3)\cdot 2xe^{x^2+1}dx = \int_{a^2+1}^{b^2+1} -3e^z dz = \left[-3e^z\right]_{a^2+1}^{b^2+1} = \left[-3\cdot e^{x^2+1}\right]_a^b = F(x)\,_a^b$

f) $\int_a^b f(x)dx = \int_a^b (-2)\cdot(-2x+1)e^{-x^2+x-1}dx = \int_{-a^2+a-1}^{-b^2+b-1} -2e^z dz = \left[-2e^z\right]_{-a^2+b-1}^{-b^2+b-1} = \left[-2\cdot e^{-x^2+x-1}\right]_a^b = F(x)\,_a^b$

g) $\int_a^b f(x)dx = \int_a^b (3x^2-2x-2)e^{x^3-x^2-2x+1}dx = \int_{a^3-a^2-2a+1}^{b^3-b^2-2b+1} e^z dz = \left[e^z\right]_{a^3-a^2-2b+1}^{b^3-b^2-2b+1} = \left[e^{x^3-x^2-2x+1}\right]_a^b = F(x)\,_a^b$

h) $\int_a^b f(x)dx = \left[(x+3)\cdot e^x\right]_a^b - \int_a^b 1\cdot e^x dx = \left[(x+3)\cdot e^x\right]_a^b - \left[e^x\right]_a^b = \left[(x+3)e^x - 1e^x\right]_a^b = \left[(x+2)\cdot e^x\right]_a^b = F(x)\,_a^b$

i) $\int_a^b f(x)dx = \left[(-2x-1)\cdot e^x\right]_a^b - \int_a^b (-2)\cdot e^x dx = \left[(-2x-1)\cdot e^x\right]_a^b - \left[(-2)e^x\right]_a^b = \left[(-2x-1)e^x - (-2)e^x\right]_a^b =$

$\left[(-2x+1)\cdot e^x\right]_a^b = F(x)\,_a^b$

j) $\int_a^b f(x)dx = \left[x^2\cdot e^x\right]_a^b - \int_a^b 2x\cdot e^x dx = \left[x^2\cdot e^x\right]_a^b - \left[\left[2xe^x\right]_a^b - \int_a^b 2e^x dx\right] = \left[x^2\cdot e^x\right]_a^b - \left[\left[2xe^x\right]_a^b - \left[2e^x\right]_a^b\right] =$

$\left[x^2e^x - 2xe^x + 2e^x\right]_a^b = \left[(x^2-2x+2)\cdot e^x\right]_a^b = F(x)\,_a^b$

k) $\int_a^b f(x)dx = \left[(x+3)\cdot(-e^{-x})\right]_a^b - \int_a^b 1\cdot(-e^{-x})dx = \left[(-x-3)\cdot e^{-x}\right]_a^b - \left[e^{-x}\right]_a^b = \left[(-x-3)e^{-x} - 1e^{-x}\right]_a^b =$

$\left[(-x-4)e^{-x}\right]_a^b = F(x)\,_a^b$

l) $\int_a^b f(x)dx = \left[(-3x+11)\cdot(\frac{1}{3}e^{3x})\right]_a^b - \int_a^b (-3)\cdot\frac{1}{3}e^{3x})dx = \left[(-x+\frac{11}{3})\cdot e^{3x}\right]_a^b - \left[-\frac{1}{3}e^{3x}\right]_a^b =$

$\left[(-x+\frac{11}{3})e^{3x} + \frac{1}{3}e^{3x}\right]_a^b = \left[(-x+4)e^{3x}\right]_a^b = F(x)\,_a^b$

m) $\int_a^b f(x)dx = \left[(-6x-1)\cdot(-\frac{1}{3}e^{-3x+1})\right]_a^b - \int_a^b (-6)\cdot(-\frac{1}{3}e^{-3x+1})dx = \left[(2x+\frac{1}{3})\cdot e^{-3x+1}\right]_a^b - \left[-\frac{2}{3}e^{-3x+1}\right]_a^b =$

$\left[(2x+\frac{1}{3})e^{-3x+1} + \frac{2}{3}e^{-3x+1}\right]_a^b = \left[(2x+1)e^{-3x+1}\right]_a^b = F(x)\,_a^b$

n) $\int_a^b f(x)dx = \left[(2x-5)\cdot\frac{1}{2}e^{2x+1}\right]_a^b - \int_a^b 2\cdot\frac{1}{2}e^{2x+1}dx = \left[(x-\frac{5}{2})\cdot e^{2x+1}\right]_a^b - \left[\frac{1}{2}e^{2x+1}\right]_a^b = \left[(x-\frac{5}{2})e^{2x+1} - \frac{1}{2}e^{2x+1}\right]_a^b =$

$\left[(x-3)e^{2x+1}\right]_a^b = F(x)\,_a^b$

o) $\int_a^b f(x)dx = \int_a^b \frac{1}{2}\cdot 2x\cdot\frac{1}{x^2+1} = \int_{a^2+1}^{b^2+1} \frac{1}{2}\cdot\frac{1}{z}dz = \left[\frac{1}{2}\ln(z)\right]_{a^2+1}^{b^2+1} = \left[\frac{1}{2}\ln(x^2+1)\right]_a^b = F(x)\,_a^b$

p) $\int_a^b f(x)dx = \int_a^b \frac{1}{2}\cdot 2\cdot\frac{1}{(2x+4)^2}dx = \int_{(2a+4)^2}^{(2b+4)^2}\frac{1}{2}\cdot\frac{1}{z^2}dz = \left[-\frac{1}{2z}\right]_{(2a+4)^2}^{(2b+4)^2} = \left[-\frac{1}{2(2x+4)}\right]_a^b = F(x)\Big|_a^b$

q) $\int_a^b f(x)dx = \int_a^b (1+\ln(x))dx = x + x\cdot\ln(x) - x\Big|_a^b = F(x)\Big|_a^b$ (with antiderivative of ln from formula booklet)

r) $\int_a^b f(x)dx = \int_a^b \ln(x)\cdot\ln(x)\,dx = \left[(x\ln(x)-x)\ln(x)\right]_a^b - \int_a^b (x\ln(x)-x)\cdot\frac{1}{x}dx = \left[x(\ln(x))^2 - x\ln(x)\right]_a^b - \int_a^b (\ln(x)-1)dx$

$= \left[x(\ln(x))^2 - x\ln(x)\right]_a^b - \; x\ln(x) - x - x\Big|_a^b = \left[x(\ln(x))^2 - 2x\ln(x) + 2x\right]_a^b = F(x)\Big|_a^b$

s) $\int_a^b f(x)dx = \int_a^b \left(1-(\ln(1-x))^2\right) dx = x\Big|_a^b - \int_a^b (\ln(1-x))^2\; dx$

$= x\Big|_a^b + \left[(1-x)(\ln(1-x))^2 - 2(1-x)\ln(1-x) + 2(1-x)\right]_a^b = \left[(1-x)(\ln(1-x))^2 - 2(1-x)\ln(1-x) + (1-x) + 1\right]_a^b$

$= \left[(1-x)((\ln(1-x))^2 - 2\ln(1-x) + 1) + 1\right]_a^b = \left[(1-x)(\ln(1-x)-1)^2 + 1\right]_a^b = \left[(1-x)(1-\ln(1-x))^2\right]_a^b = F(x)\Big|_a^b$

(The 1 cancels out when taking differences!)

NR: $-\int_a^b (\ln(1-x))^2\; dx = \int_{1-a}^{1-b} \ln z\cdot\ln z\; dz = (z\ln z - z)\ln z\Big|_{1-a}^{1-b} - \int_{1-a}^{1-b}(z\ln z - z)\cdot\frac{1}{z}dz$

$= \left[z(\ln z)^2 - z\ln z\right]_{1-a}^{1-b} - \int_{1-a}^{1-b}(\ln z - 1)dz = \left[z(\ln z)^2 - z\ln z\right]_{1-a}^{1-b} - \; z\ln z - z - z\Big|_{1-a}^{1-b} = \left[z(\ln z)^2 - 2z\ln z + 2z\right]_{1-a}^{1-b}$

$= \left[(1-x)(\ln(1-x))^2 - 2(1-x)\ln(1-x) + 2(1-x)\right]_a^b$ mit $z(x) = 1 - x$

Exercise 11: improper integrals

a) $\int_1^\infty \frac{1}{x^2}dx = 1$

b) $\int_0^\infty \frac{1}{(x+1)^2}dx = 1$

c) $\int_1^\infty \frac{1}{x^{1,5}}dx = 2$

d) $\int_0^1 \frac{1}{x^{0,5}}dx = 2$

e) $\int_{-1}^0 \frac{1}{\sqrt{x+1}}dx = 2$

f) $\int_0^\infty e^{-x}dx = 1$

g) $\int_0^\infty x\cdot e^{-x^2}dx = \frac{1}{2}$

h) $\int_0^1 \frac{1}{x^2}\cdot e^{-\frac{1}{x}}dx = \frac{1}{e}$

Exercise 12: improper integrals

For n < 1 its A = $\int_0^1 \frac{1}{x^n}dx = \left[\frac{1}{1-n}x^{1-n}\right]_0^1 = \frac{1}{1-n}$ and

for n > 1 its A = $\int_1^\infty \frac{1}{x^n}dx = \lim_{b\to\infty}\left[\frac{1}{1-n}x^{1-n}\right]_0^b = \frac{1}{n-1}$.