5.5. Integration

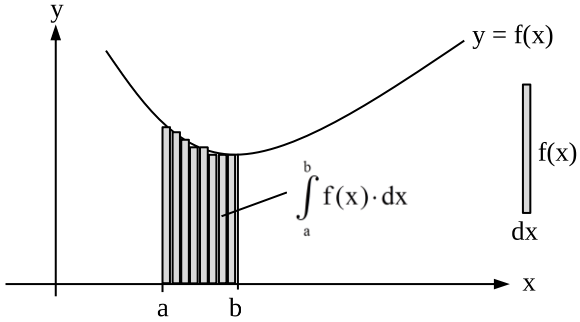
5.5.1. Graphical integration

Definition:
The **oriented Measure** of the area bounded by

- the x-Axis,
- the curve y = f(x),
- the **lower limit** x = a and
- the **upper limit** x = b

is called the **Integral** $\int_a^b f(x)dx$

from a to b of f (x) with respect to x.



Remarks:
1. The **oriented measure** has a **positive** resp. **negative sign**, if the area is above resp. below the x-axis.
2. The integral can be **approximated graphically** by taking **sums** $\sum_{i=1}^{n} f(x_i)\cdot\Delta x$ of n rectangle strips with heights $f(x_1)$, $f(x_2)$, ... and width $\Delta x = \frac{b-a}{n}$ between the limits a and b. For $n \to \infty$ the strips become ever narrower ($\Delta x \to 0$) and the sums S_n tend to the Integral: $\sum_{i=1}^{n} f(x_i)\cdot\Delta x \to \int_a^b f(x)\cdot dx$ for $n \to \infty$

Exercises on graphical integration

5.5.2. Antiderivatives

Definition:
A function $F_c(x) = F_0(x) + c$ with the **integration constant c** is the **antiderivative** of the function f(x), if $F_c'(x) = F_0'(x) = f(x)$.
Since the integration constant c vanishes in the process of differentiating, it can assume any real value: $c \in \mathbb{R}$. We determine the antiderivative by **integrating** f.

Integration rules:

	function f(x) =	Antiderivative $F_c(x)$ =
constant factors stay	$a\cdot g(x)$	$a\cdot G_c(x)$
sums are integrated separately	g(x) + h(x)	$G_c(x) + H_c(x)$
power functions	x^n with $n \in \mathbb{R}\backslash\{-1\}$	$\frac{1}{n+1}x^{n+1} + c$
	x^{-1}	ln(x) + c
polynomial functions	$a_n{}^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$	$\frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + ... + \frac{a_1}{2}x^2 + a_0x + c$

Exercises on integration No 1

5.5.3. The fundamental theorem of calculus

Fundamental theorem of calculus
The integral of the function f(x) with respect to x between a and b can be obtained by taking the difference of the antiderivative F(x) at the limits a and b.

$$\int_a^b f(x)dx \;=\; F(x)\Big|_a^b \;=\; F(b) - F(a)$$

Hence the antiderivative F(x) measures the area under the curve y = f(x) and is therefore written $F(x) = \int f(x)dx$. It is also called an **indefinite integral** (i.e. without given limits).

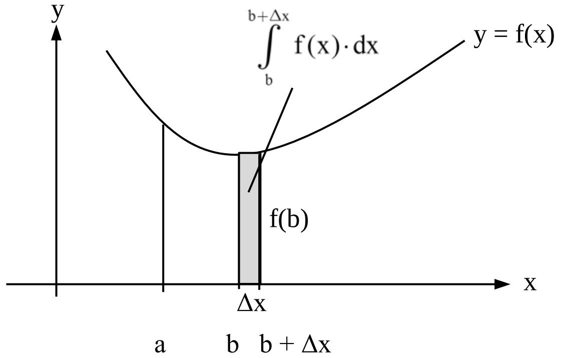
Proof:
The rate of change of the integral at x = b is

$$\left(\int_a^b f(x)dx\right)' = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_a^{b+\Delta x} f(x)dx - \int_a^b f(x)dx\right)$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_b^{b+\Delta x} f(x)dx$$

$$= \frac{1}{dx}\cdot f(b)\cdot dx$$

$$= f(b).$$



So the derivative of the integral is the function f itself. Hence the integral is the antiderivative of f: $\int_a^b f(x)\cdot dx = F_0(b) + c$. We can obtain the integration constant c by regarding the special case b = a with $0 = \int_a^a f(x)\cdot dx = F_0(a) + c \Leftrightarrow c = -F_0(a)$. Substituting c in the general case yields $\int_a^b f(x)\cdot dx = F_0(b) - F_0(a)$, qed.

Example:

$$\int_1^2 (x^2+5)dx = \left[\frac{1}{3}x^3+5x\right]_1^2 = (\frac{1}{3}2^3+5\cdot 2) - (\frac{1}{3}1^3+5\cdot 1) = \frac{22}{3}$$

Exercises on integration No. 2 a) – c)

5.5.4. Properties of the integral

Additivity

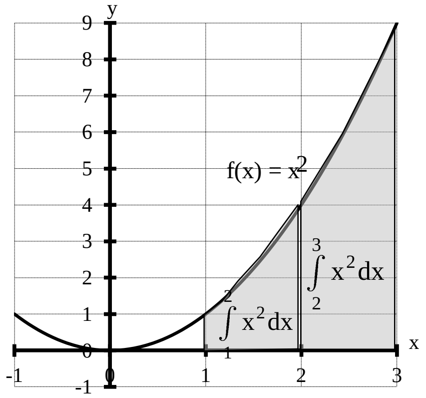
Example: Exercises on integration No 2d)

a) $\int_1^2 x^2dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{7}{3}$

b) $\int_2^3 x^2dx = \left[\frac{1}{3}x^3\right]_2^3 = \frac{19}{3}$

c) $\int_1^3 x^2dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{26}{3}$

$$= \int_1^2 x^2dx + \int_2^3 x^2dx .$$



Additivity of Intervals
The Integral over [a;c] can be split into a sum of integrals over partial intervals [a;b] and [b;c]:

$$\int_a^c f(x)dx \;=\; \int_a^b f(x)dx + \int_b^c f(x)dx$$

Especially areas above and below the x-axis must be integrated separatedly to avoid cancelling each other out.

Exchange of limits

Example: Exercises on integration No 2e)

$$\int_1^2 x^2 dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \text{ aber } \int_2^1 x^2 dx = \left[\frac{1}{3}x^3\right]_2^1 = \frac{1}{3} - \frac{8}{3} = -\frac{7}{3}$$

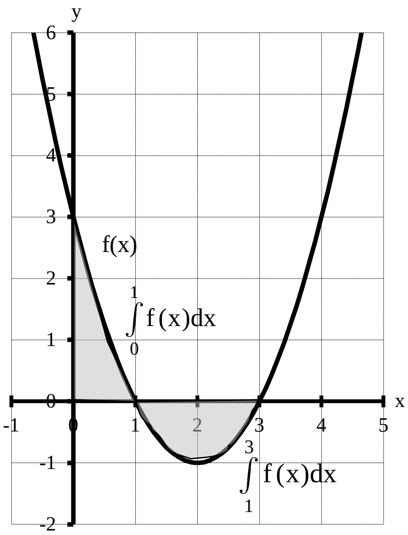
Exchange of limits
Exchanging limits resp. integration backwards results in **sign change** because of **negative strip width dx < 0**:

$$\int_b^a f(x)dx = F(a) - F(b) = -(F(b) - F(a)) = -\int_a^b f(x)dx\,.$$

Areas below the x-axis

Example: Exercises on integration No 2f)

$$A = \left|\int_0^1 (x^2 - 4x + 3)dx\right| + \left|\int_1^3 (x^2 - 4x + 3)dx\right| = \left|\frac{4}{3}\right| + \left|-\frac{4}{3}\right| = \frac{8}{3} \text{ FE}$$



Areas below the x-axis
Areas below the x-axis result in **sign change** because of **negative strip height** f(x) < 0:

$$\int_a^b f(x)dx < 0, \text{ for } f(x) \le 0 \text{ on } [a; b].$$

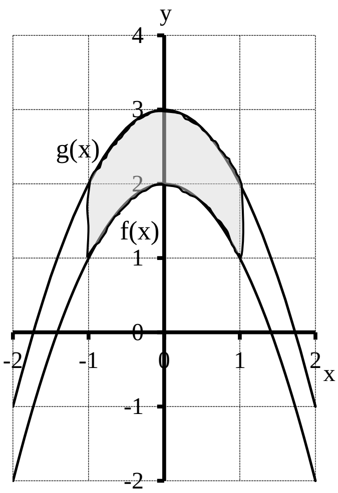
Consequently areas below the x-axis must be integrated **separately** and their negative sign has to be removed by taking the **absolute value.**

Exercises on integration Nos 3 and 4

Areas between two curves

Example: Exercises on integration No 5a)

$$A = \int_{-1}^{1} (g(x) - f(x))dx = \int_{-1}^{1} 1dx = x\Big|_{-1}^{1} = 2$$



Areas between two curves
The area between two functions f(x) ≥ g(x) can be approximated as sum of strips with height f(x) – g(x). Hence its exact value is:

$$A = \int_a^b f(x) - g(x)\, dx$$

The integral must be **split up at intersection points** to avoid negative values.

For areas with $f \ge g$ it is $\int_a^b f(x) - g(x)\,dx$ and for $g \ge f$ it is $\int_a^b g(x) - f(x)\,dx$.

Exercises on integration No 5 - 7

5.5.5. The substitution method

Theorem: substitution method for integration or inverse of the chain rule

$$\int_a^b f(z(x))\cdot z'(x)dx = \int_a^b f(z(x))\cdot\frac{dz}{dx}\cdot dx = \int_{z(a)}^{z(b)} f(z)dz$$

Examples : $\int_a^b z'(x)\cdot f(z(x))dx = \int_{z(a)}^{z(b)} f(z)dz$ **:**

1. $\int_1^2 2x\cdot e^{x^2}dx = \int_1^4 e^z dz = e^4 - e^1$ with $z(x) = x^2$ and $f(z) = e^z$.
2. $\int_0^1 x\cdot e^{x^2+5}dx = \frac{1}{2}\int_0^1 2x\cdot e^{x^2+5}dx = \int_5^6 e^z dz = e^6 - e^5$ with $z(x) = x^2 + 5$ and $f(z) = e^z$
3. $\int_2^3 \frac{3x}{x^2-1}dx = \frac{3}{2}\int_2^3 2x\cdot\frac{1}{x^2-1}dx = \frac{3}{2}\int_3^8 \frac{1}{z}dz = \frac{3}{2}(\ln(8) - \ln(3)) = \frac{3}{2}\ln\left(\frac{8}{3}\right)$ with $z(x) = x^2 - 1$ and $f(z) = \frac{1}{z}$

Exercises on integration No 8

5.5.6. Partial integration

Theorem: Product rule for integration or integration by parts

$$\int_a^b g'(x)\cdot h(x)\ dx = \left[g(x)\cdot h(x)\right]_a^b - \int_a^b g(x)\cdot h'(x)\ dx$$

Proof

The product rule for differentiation with reversed sides reads

$$g'(x)\cdot h(x) + g(x)\cdot h'(x) = \left(g(x)\cdot h(x)\right)' =$$

Integrating on both sides yields:

$$\int_a^b g'(x)\cdot h(x)\ dx + \int_a^b g(x)\cdot h'(x)\ dx = \int_a^b \left(g(x)\cdot h(x)\right)'dx$$

$$= \left[g(x)\cdot h(x)\right]_a^b \text{ qed.}$$

Examples: $\int_a^b g'(x)\cdot h(x)\ dx = \left[g(x)\cdot h(x)\right]_a^b - \int_a^b g(x)\cdot h'(x)\ dx$

1. $\int_0^1 x\cdot e^x\ dx = \int_0^1 e^x\cdot x\ dx = \left[e^x\cdot x\right]_0^1 - \int_0^1 e^x\cdot 1\ dx = \left[e^x\cdot x\right]_0^1 - \left[e^x\right]_0^1 = \left[e^x\cdot(x-1)\right]_0^1 = 1$
2. $\int_0^\pi (x+\pi)\cdot\cos x\ dx = \int_0^\pi (\cos x)\cdot(x+\pi)\ dx = \left[(\sin x)\cdot(x+\pi)\right]_0^\pi - \int_0^\pi (\sin x)\cdot 1\ dx = \left[(\sin x)\cdot(x+\pi)\right]_0^\pi - \left[-\cos x\right]_0^\pi = \left[(\sin x)\cdot(x+\pi) - \cos x\right]_0^\pi = 0$
3. $\int_1^2 \ln x\ dx = \int_1^2 1\cdot\ln x\ dx = \left[x\cdot\ln x\right]_1^2 - \int_1^2\left(x\cdot\frac{1}{x}\right)dx = \left[x\cdot\ln x\right]_1^2 - \left[x\right]_1^2 = \left[x\cdot\ln x - x\right]_1^2 = 2\ln 2 - 1$

Exercises on integration Nos 9 and 10

5.5.7. Partial fraction decomposition

Theorem: Decomposition of rational functions into partial fractions

For $a, b, x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$ holds $\frac{ax+b}{(x-x_1)\cdot(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$ with $A = \frac{ax_1+b}{x_1-x_2}$ and $B = \frac{ax_2+b}{x_2-x_1}$

Proof:

$\frac{A}{x-x_1} + \frac{B}{x-x_2} = \frac{A\cdot(x-x_2)+B\cdot(x-x_1)}{(x-x_1)\cdot(x-x_2)} = \frac{(A+B)\cdot x - Ax_2 - Bx_1}{(x-x_1)\cdot(x-x_2)} = \frac{ax+b}{(x-x_1)\cdot(x-x_2)} \Leftrightarrow (A+B)\cdot x - Ax_1 - Bx_2 =$ $ax + b\ \forall x \in \mathbb{R} \Leftrightarrow A + B = a$ and $-Ax_1 - Bx_2 = b \Leftrightarrow A = \frac{ax_1+b}{x_1-x_2}$ and $B = \frac{ax_2+b}{x_2-x_1}$.

Example: $\frac{ax+b}{(x-x_1)\cdot(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$ with $A = \frac{ax_1+b}{x_1-x_2}$ and $B = \frac{ax_2+b}{x_2-x_1}$

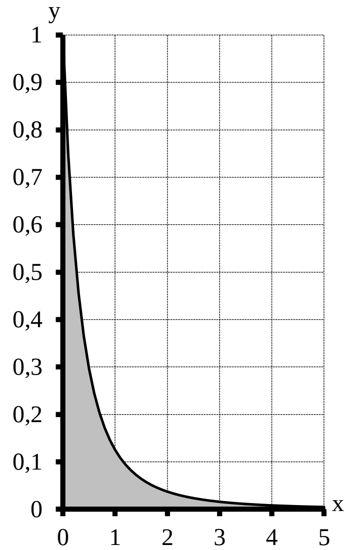
$$\int_2^3 \frac{2}{x^2-1}dx = \int_2^3 \frac{2}{(x+1)(x-1)}dx = \int_2^3 \frac{1}{x+1}dx - \int_2^3 \frac{1}{x-1}dx = \ln(x+1)\Big|_2^3 - \ln(x-1)\Big|_2^3 = \ln\frac{4}{3} - \ln\frac{1}{2} = \ln\frac{8}{3}$$

5.5.8. Improper integrals

Definition

If a curve converges for $x \to \pm\infty$ or $x \to a$ **with sufficient speed** to its **asymptote** the measure of the enclosed area also converges to a finite limit although the limit area has infinite range and no closed circumference. Such limits of definite integrals are called **improper integrals** and can be written as follows:

$$\int_a^\infty f(x)dx := \lim_{b\to\infty}\int_a^b f(x)dx \quad \text{resp.} \quad \int_{-\infty}^b f(x)dx := \lim_{a\to-\infty}\int_a^b f(x)dx$$



Example:

$$\int_0^\infty \frac{1}{(x+1)^3}dx = \lim_{b\to\infty}\int_0^b \frac{1}{(x+1)^3}dx = \lim_{b\to\infty}\left[\frac{-2}{(x+1)^2}\right]_0^b = \lim_{b\to\infty}\left[\frac{-2}{(b+1)^2}+2\right] = 2$$

Exercises on integration Nos 11 and 12

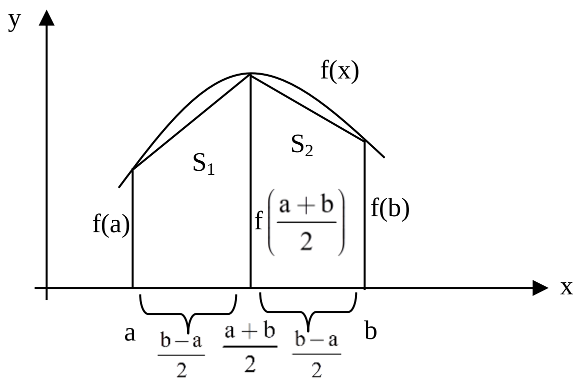
5.5.9. Numerical integration

Linear approximation with the long trapeziodal rule

$$\int_a^b f(x)dx \approx S_1 + S_2$$

$$= \frac{b-a}{2}\cdot\frac{f(a)+f\left(\frac{a+b}{2}\right)}{2} + \frac{b-a}{2}\cdot\frac{f\left(\frac{a+b}{2}\right)+f(b)}{2}$$

$$= \frac{b-a}{4}\left[f(a) + 2\cdot f\left(\frac{a+b}{2}\right) + f(b)\right]$$



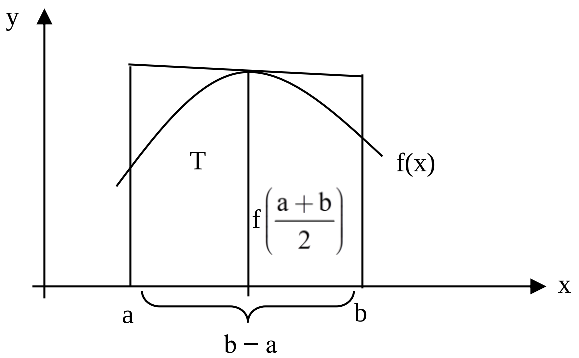
Quadratic approximation with the simpson rule

$$\int_a^b f(x)dx \approx \frac{2(S_1+S_2)+T}{3}$$

$$= \frac{2}{3}(S_1 + S_2) + \frac{1}{3}T$$

$$= \frac{b-a}{6}\left[f(a) + 2\cdot f\left(\frac{a+b}{2}\right) + f(b)\right] + \frac{1}{3}(b-a)\cdot f\left(\frac{a+b}{2}\right)$$

$$= \frac{b-a}{6}\left[f(a) + 4\cdot f\left(\frac{a+b}{2}\right) + f(b)\right]$$



Examples:

1. Linear approximation with the long trapezoidal rule: $\int_1^2 x^3 dx \approx \frac{2-1}{4}[1^3 + 2\cdot\left(\frac{1+2}{2}\right)^3 + 2^3] = \frac{63}{16} = 3{,}9375$

2. Quadratic approximation with the simpson rule: $\int_1^2 x^3 dx \approx \frac{2-1}{6}[1^3 + 4\cdot\left(\frac{1+2}{2}\right)^3 + 2^3] = \frac{15}{4} = 3{,}75$ (!)

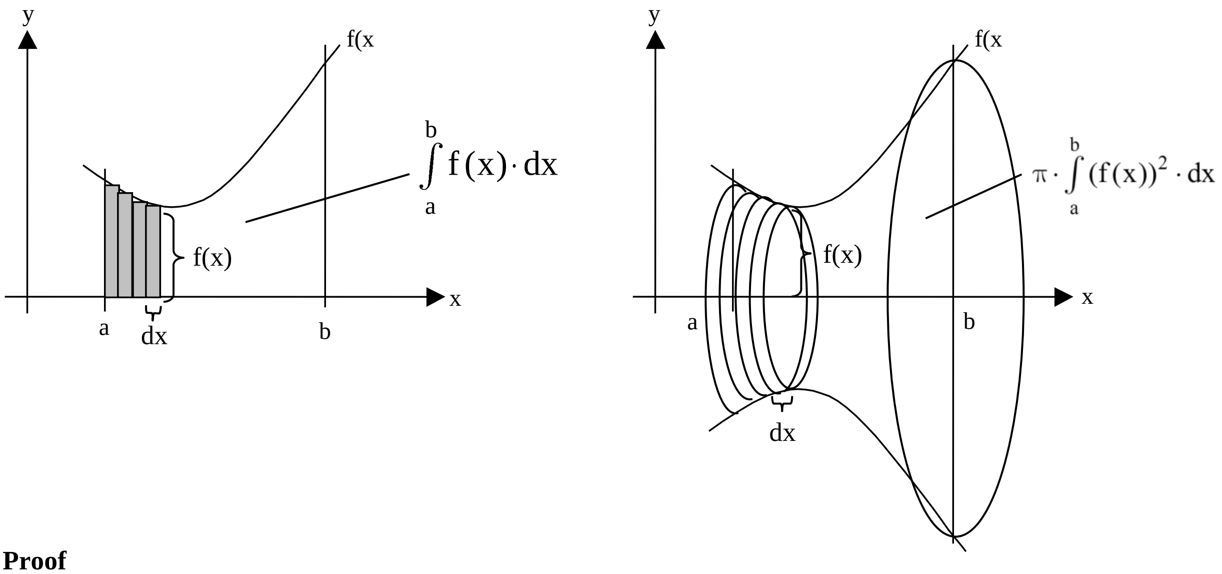
3. Exact vale with the fundamental theorem (using the antiderivative): $\int_1^2 x^3 dx = \left[\frac{1}{4}x^4\right]_1^2 = \frac{15}{4} = 3{,}75$

Applications of integration No 1

5.5.10. Volumes of rotation bodies

Theorem (Rotation about the x-axis)

The body defined by the rotation of y = f(x) about the x-axis in the interval [a;b] has the volume V = $\pi\cdot\int_a^b (f(x))^2\cdot dx$.



Proof

The area $\int_a^b f(x)dx$ between the curve y = f(x) and the x-axis can be approximated by taking the sums **S** of measures **f(x)·dx** of **rectangles** with **height** f(x) und **width** dx between the limits **a** and **b**. If the rectangles are replaced by cylinders $\pi\cdot f(x))^2\cdot dx$ with **radius** f(x) and **height** dx, we get the volume V = $\pi\cdot\int_a^b (f(x))^2\cdot dx$ enclosed by the rotating curve.

Theorem (Rotation about the y-axis)
The body define by the rotation of the monotone function f(x) with inverse $f^{-1}(x)$ about the y-axis in the interval [f(a); f(b)] has the volume V = $\pi\cdot\int_{f(a)}^{f(b)} (f^{-1}(y))^2\cdot dy$.
Proof: obvious

Exercises: Applications of integration Nr. 2 - 5