5.6. Exercises on differential equations

Exercise 1
Find the particular solution of the following differential equations, giving your answer in the form y = f(x) simplified as far as possible.

a) $\frac{dy}{dx} = \frac{2x^2}{3y}$, y = 0 when x = 0 b) $\frac{dy}{dx} = 4xy^2$, y = 1 when x = 0 c) $\frac{dy}{dx} = \frac{4y}{x}$, y = 2 when x = 1

d) $\frac{dy}{dx} = -3x^2y$, y = 3 when x = 0 e) $\frac{dy}{dx} = \frac{\sin(x)}{\cos(y)}$, y = 0 when x = $\frac{\pi}{3}$ f) $\frac{dy}{dx} = \frac{\sec^2(x)}{\sec^2(y)}$, y = 0 when x = $\frac{\pi}{3}$

g) $2(1 + x)\frac{dy}{dx} = 1 + y^2$, y(0) = 0 h) $(1 + x^2)\frac{dy}{dx} = 2x\sqrt{1-y^2}$, y(0) = 0 i) $\frac{dy}{dx} = 2e^{x + 2y}$, y = 0 when x = 0

j) $\frac{dy}{dx} = 2y(1 - x)$, y(1) = 1 k) $\frac{dN}{dt} = -kN$, $N(0) = N_0$ l) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$, $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$

Exercise 2
Find the general solution of the following differential equations, giving your answer in the form y = f(x) simplified as far as possible.

a) $2y\frac{dy}{dx} = 3x^2$ b) $\frac{1}{y^2}\frac{dy}{dx} = 2x$ c) $x\frac{dy}{dx} = \sec(y)$ d) $\csc(x)\cdot\frac{dy}{dx} = 1 + y^2$ e) $(x - 1)\frac{dy}{dx} = x(y + 3)$

Exercise 3
Find the general solution of the following differential equations. You may leave your answers in implicit form.

a) $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - 2$ b) $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - \frac{4y}{x}$ c) $\frac{dy}{dx} = \frac{x+2y}{x}$ d) $\frac{dy}{dx} = \frac{xy-y^2}{x^2}$

e) $x\frac{dy}{dx} = 3x - 4y$ f) $x\frac{dy}{dx} = 2xy + y^2$ g) $\frac{dy}{dx} = \frac{y}{x} + e^{-\frac{y}{x}}$ h) $\frac{dy}{dx} = \frac{1}{\cos\left(\frac{y}{x}\right)} + \frac{y}{x}$

Exercise 4
Find the particular solution of the following differential equations for the given conditions.

a) $\frac{dy}{dx} = \frac{x+y}{x}$, y = 3 when x = e b) $\frac{dy}{dx} = \frac{x^2+y^2+xy}{x^2}$, y(1) = 1 c) $\frac{dy}{dx} = x^2 + y^2$, y = 4 when x = 1

Exercise 5
A particular solution of $x\cdot\frac{dv}{dx} = f(v)$ has v = 2 when x = e. If k is the value of v when x = 1, show that $\int_k^2 \frac{1}{f(y)}dy = 1$.

Exercise 6
a) The differential equation $\frac{dy}{dx} = \ln(y) - \ln(x)$ has a particular solution with x = e, y = 2e. Show that $\int_k^2 \frac{1}{\ln(v)-v}dv = 1$

b) Find the value of k to three significant figures.

Exercise 7
Use an integrating factor to find the general solution to each of the following linear differential equations.

a) $\frac{dy}{dx} + 2y = e^x$ b) $\frac{dy}{dx} - 4y = e^x$ c) $\frac{dy}{dx} + y\cdot\cot(x) = 1$ d) $\frac{dy}{dx} - y\cdot\tan(x) = \sec(x)$

e) $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ f) $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^3}$ g) $x^2\frac{dy}{dx} - 2xy = \frac{x^4}{x-3}$ h) $\frac{dy}{dx} + y\cdot\sin(x) = e^{\cos(x)}$.

Exercise 8
Use an integrating factor to find the particular solution to each of the following linear differential equations.

a) $\frac{dy}{dx} + y = e^x$ for y(1) = e b) $x^2\frac{dy}{dx} + xy = \frac{2}{x}$ through (1|1) c) $\cos(x)\cdot\frac{dy}{dx} + y\cdot\sin(x) = \cos^2(x)$ with y(0) = 2

Exercise 9

Use Euler's method with step length 0,1 to find an approximate value of y when x = 0,4 for the following differential equations.

a) $\frac{dy}{dx} = x^2 - y^2$ for y(0) = 1 b) $\frac{dy}{dx} = \ln(x + y)$ for y(0) = 2 c) $\frac{dy}{dx} - y^2 = \sin(x^2)$ with y(0) = 1

d) $\frac{dy}{dx} - y = 2e^x$ for y(0) = 0 e) $(x + y)\frac{dy}{dx} = 3x^2 + y^2$ for y(0) = 2 f) $(x + y)\frac{dy}{dx} = e^{x + 0,2y}$ with y(0) = 1

Exercise 10

For the following differential equations, find the equations for the isoclines and hence by sketching these construct he slope field at the points (x|y) where x,y ∈ {−2; −1; 0; 1; 2}. Then sketch the solution curves passing through the points (−2|1) and (−2|−1). In e) start with (−1|2) instead.

a) $\frac{dy}{dx} = 2x - y$ b) $\frac{dy}{dx} = \frac{x}{y}$ c) $\frac{dy}{dx} = xy$

d) $\frac{dy}{dx}$ xy + 2x e) $\frac{dy}{dx} = 2x - y^2$ f) $\frac{dy}{dx} = x^2 + y - 3$

Exercise 11

Find a Taylor polynomial about the point x_0 that approximates the solution of the following differential equations.

a) $\frac{dy}{dx} = y^2 - x$, $y_0 = 1$ at $x_0 = 0$ b) $(1 + 2x)\frac{dy}{dx} = x + 4y^2$, $y_0 = \frac{1}{2}$ at $x_0 = 0$

c) $\frac{dy}{dx} = \cos(x) - \sin(x) + x^2$, $y_0 = \frac{\pi}{2}$ at $x_0 = -\pi$ d) $\sin(x)\frac{dy}{dx} + y\cdot\cos(x) = y^2$, $y_0 = \sqrt{2}$ at $x_0 = \frac{\pi}{4}$

5.6. Solutions on the exercises on differential equations

Exercise 1

a) $y = \frac{2}{3}x^{1,5}$ b) $y = \frac{1}{1-2x^2}$ c) $y = 2x^4$ d) $y = 3\,e^{-x^2}$ e) $y = \arcsin\left(\frac{1}{2} - \cos(x)\right)$ f) $y = \arctan\ \tan(x) - \sqrt{3}$

g) $y = \tan\ \ln\ \sqrt{1+x}$ h) $y = \sin\ \ln\ 1+x^2$ i) $y = -\frac{1}{2}\ln\ 5-4e^x$ j) $y = e^{-(1-x)^2}$ k) $N = N_0 \cdot e^{-kt}$ l) $y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}$

Exercise 2

a) $y = \pm\sqrt{x^3+c}$ b) $y = -\frac{1}{x^2+c}$ c) $y = \arcsin\ \ln|x|+c$ d) $y = \tan(c - \cos(x))$ e) $y = Ae^x(x-1) - 3$

Exercise 3

a) $\frac{1}{3}\ln\left|\frac{y-2x}{y+x}\right| = \ln|x| + c$ b) $\frac{1}{5}\ln\left|\frac{y-5x}{y}\right| = \ln|x| + c$ c) $\ln\left|\frac{y}{x}+1\right| = \ln|x| + c$ d) $\frac{x}{y} = \ln|x| + c$

e) $-\frac{1}{5}\ln\left|3-\frac{5y}{x}\right| = \ln|x| + c$ f) $\ln\left|\frac{y}{x}\left(1-\frac{y}{x}\right)\right| = \ln|x| + c$ g) $e^{\frac{y}{x}} = \ln|x| + c$ h) $\sin\left(\frac{y}{x}\right) = \ln|x| + c$

Exercise 4

a) $y = x \cdot \ln|x| + \left(\frac{3-e}{e}\right) \cdot x$ b) $y = x \cdot \tan\left(\ln|x| + \frac{\pi}{4}\right)$ c) $y = x \cdot \sqrt{2 \cdot \ln|x| + 16}$

Exercise 5

$$\int_k^2 \frac{1}{f(y)}dy = \int_1^e \frac{1}{x}dx = \ln|x|\Big|_1^e = \ln(e) - \ln(1) = 1$$

Exercise 6

a) $\frac{dy}{dx} = \ln\left(\frac{y}{x}\right) \Leftrightarrow v + x\frac{dv}{dx} = \ln(v) \Leftrightarrow x\frac{dv}{dx} = \ln(v) - v \Leftrightarrow \int_k^2 \frac{1}{\ln(v)-v}dv = \int_e^{2e}\frac{1}{x}dx = \ln|x|\Big|_e^{2e} = 2 - 1 = 1.$

b) Approximation with GDC yields $k \approx 3{,}79$.

Exercise 7

a) $y = \frac{1}{3}e^x + c \cdot e^{-2x}$ b) $y = -\frac{1}{3}e^x + c \cdot e^{4x}$ c) $y = -\cot(x) + \csc(x)$ d) $y = \frac{x+c}{\cos(x)}$

e) $y = \frac{\ln|x|+c}{x}$ f) $y = -\frac{1}{x^2} + \frac{c}{x}$ g) $y = x^2(\ln|x-3| + c)$ h) $y = e^{\cos(x)} \cdot (x + c)$

Exercise 8

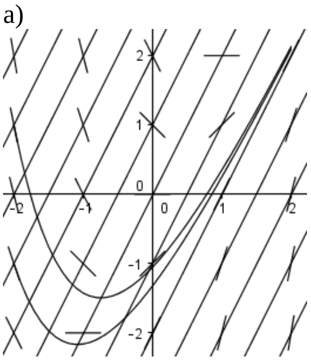
a) $y = \frac{1}{2}(e^x + e^{2-x})$ b) $y = -\frac{2}{x^2} + \frac{3}{x}$ c) $y = (x + 2) \cdot \cos(x)$

Exercise 9

a) $y(0{,}4) \approx 0{,}708$ b) $y(0{,}4) \approx 2{,}32$ c) $y(0{,}4) \approx 1{,}57$ d) $y(0{,}4) \approx 1{,}07$ e) $y(0{,}4) \approx 2{,}89$ f) $y(0{,}4) \approx 1{,}45$

Exercise 10

a)



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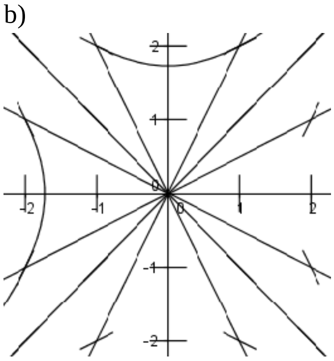

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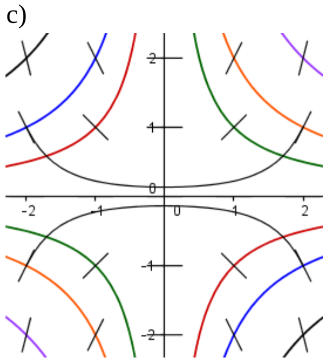

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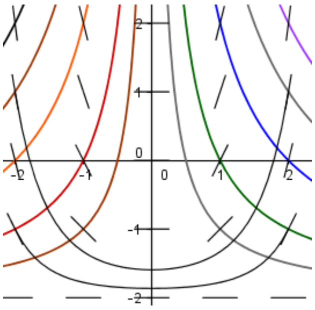

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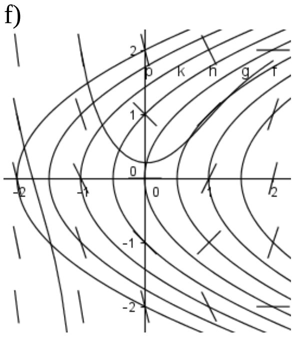

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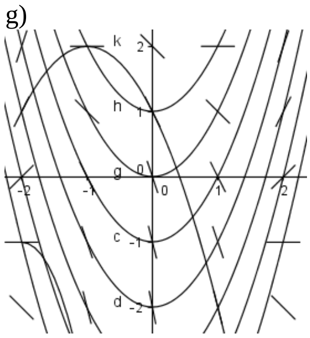

Exercise 11

a) $y = 1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3$

b) $y = \frac{1}{2} + x + \frac{3}{2}x^2 + \frac{4}{3}x^3$

c) $y = \frac{\pi}{2} + (\pi^2 + 2)(x + \pi) - \pi(x + \pi)^2 + \frac{1}{6}(3 + 4\pi)^2(x + \pi)^3$

d) $y = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3$