5.6. Differential equations

The relationship between cause and effect in physical phenomena can often be formulated using **differential equations** which describe how a physical measure y(x) and its derivative y'(x) depend on each other. Examples are

- **distance** s(t) and **velocity** v(t) = s'(t)
- **velocity** v(t) and **acceleration** a(t) = v'(t)
- **electrical charge** Q(t) and **amperage** I(t) = Q'(t)
- **work** W(t) und **power** P(t) = W'(t)

When y(x) and y'(x) appear together in a single equation it is generally not possible to solve the equation to y(x). However most physical applications lead to certain simple types of differential equations which can be solved with the special methods explained below. In the general case only approximate solutions are possible and two common algorithms for these will be presented too.

5.6.1. Separation of variables

Separation of variables

A differential equation of the form $\frac{dy}{dx} = f(x)\cdot g(y)$ with **initial value** $y(x_0) = y_0$ is called **variables separable** and **can often** be solved in the following way:

1. **Separation of variables**: Get all expressions in x on one side and all expressions in y on the other side: $\frac{dy}{g(y)} = f(x)\cdot dx$.
2. **Integrate** starting from the given initial value $y(x_0) = y_0$ as lower bound: $\int_{y_0}^{y} \frac{dy}{g(y)} = \int_{x_0}^{x} f(x)\cdot dx$.
3. **Solve** to y.

Caution: Steps 2 and 3 may not always work!

Example:

Find the solution y(x) to the differential equation y'(x) = 2·y(x) − 5 with initial value $y(0) = \frac{3}{2}$

Solution:

$$y'(x) = 2\cdot y(x) - 5 \qquad \text{| use differential notation}$$

$$\frac{dy}{dx} = 2\cdot y - 5 \qquad \text{| separation of variables}$$

$$\frac{dy}{2y-5} = dx \qquad \text{| integrate on both sides:}$$

$$\int_{1,5}^{y} \frac{1}{2y-5}dy = \int_{0}^{x} 1dx \qquad \text{| Solve integrals}$$

$$\left[\frac{1}{2}\ln(2y-5)\right]_{1,5}^{y} = x\,\Big|_{0}^{x} \qquad \text{| plug in}$$

$$\frac{1}{2}[\ln(2y-5) - \ln(2\cdot 3 - 5)] = x - 0 \qquad \text{| simplify: ln(1) = 0}$$

$$\ln\ 2y-5 = 2x \qquad \text{| put to the power of e}$$

$$2y - 5 = e^{2x} \qquad \text{|·simplify}$$

$$y(x) = \frac{1}{2}\ e^{2x} - 5$$

Exercises on differential equations Nos 1 and 2

5.6.2. Homogenous differential equations

A differential equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ is called **homogenous** and can be converted into a **variables separable** form using the **substitution** $v = \frac{y}{x}$ with $\frac{dy}{dx} = \frac{d(v\cdot x)}{dx} = x\cdot\frac{dv}{dx} + v\cdot\frac{dx}{dx} = x\cdot\frac{dv}{dx} + v$ according to the **product rule**. The differential equation with respect to v then becomes $x\cdot\frac{dv}{dx} + v = f(v) \Leftrightarrow \frac{dv}{dx} = \frac{f(v)-v}{x}$ and can be solved by separating the variables.

Example:

Find the general solution y(x) to the differential equation $y'(x) = \frac{x^2+y^2}{2xy}$ with $x, y > 0$.

Solution:

The differential equation is homogenous with right hand side $f\left(\frac{y}{x}\right) = \frac{x^2+y^2}{2xy} = \frac{1+\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$.

The usual substitution $v = \frac{y}{x}$ with $\frac{dy}{dx} = x\cdot\frac{dv}{dx} + v$ and $f(v) = \frac{1+v^2}{2v}$ yields the variables separable form

$$\frac{dv}{dx} = \frac{f(v)-v}{x} = \frac{1-v^2}{2vx} \quad | \text{ separation of variables}$$

$$\frac{2v}{1-v^2}\cdot dv = \frac{dx}{x} \quad | \text{ integration with indefinite lower limits} > 0$$

$$-\ln|1-v^2| = \ln(x) + c \quad | \text{ put to the power of e}$$

$$\frac{1}{1-v^2} = x\cdot e^c \quad | \text{ solve to v}$$

$$v = \sqrt{1-\frac{1}{x\cdot e^c}} \quad | \text{ resubstitute } v = \frac{y}{x} \text{ and multiply with x}$$

$$y = \sqrt{x^2-\frac{x}{e^c}}$$

Exercises on differential equations Nos 3 – 6

5.6.3. Linear differential equations

A differential equation of the form $\frac{dy}{dx} + P(x)\cdot y(x) = Q(x)$ is called **linear** and can be solved by multiplying with the **integrating factor** $I(x) = e^{\int P(x)dx}$. Since this integrating factor satisfies $\frac{dI}{dx} = I(x)\cdot P(x)$ we obtain

$$I(x)\cdot\frac{dy}{dx} + I(x)\cdot P(x)\cdot y(x) = I(x)\cdot Q(x) \quad | \text{ plug in } \frac{dI}{dx} = I(x)\cdot P(x)$$

$$I(x)\cdot\frac{dy}{dx} + \frac{dI}{dx}\cdot y(x) = I(x)\cdot Q(x) \quad | \text{ apply product rule}$$

$$\frac{d(I\cdot y)}{dx} = I(x)\cdot P(x) \quad | \text{ integrate}$$

$$I(x)\cdot y(x) = \int I(x)\cdot P(x)\cdot dx \quad | \text{ solve to y(x)}$$

$$y(x) = \frac{1}{I(x)}\int I(x)\cdot P(x)\cdot dx\,.$$

Example:
Find the solution y(x) to the differential equation $y'\cdot\cos(x) + y\cdot\sin(x) = (\cos(x))^2$ with $y(0) = 2$.

Solution:
Dividing by cos(x) yields a linear differential equation

$$y'(x) + \tan(x)\cdot y(x) = \cos(x)$$

with integrating factor

$$I(x) = e^{\int P(x)dx} = e^{\int \tan(x)dx} = \frac{1}{\cos(x)}.$$

Note that the frequently occurring integral

$$\int \tan(x)dx = -\int \frac{1}{\cos(x)}(-\sin(x))\cdot dx = -\int \frac{1}{y}dy = -\ln|y| = -\ln|\cos(x)| = \ln\left|\frac{1}{\cos(x)}\right|$$

is **not included in the formular booklet** (!)

Multiplying with I(x) obtains

$$\frac{1}{\cos(x)}\cdot y'(x) + \frac{\sin(x)}{(\cos(x))^2}\cdot y(x) = 1 \qquad | \text{ product rule}$$

$$\frac{d\left(\frac{1}{\cos(x)}\cdot y(x)\right)}{dx} = 1 \qquad | \text{ integration}$$

$$\frac{y(x)}{\cos(x)} = x + c \qquad | \text{ solve to y}$$

$$y(x) = \cos(x)\cdot(x + c) \qquad | \text{ plug in initial value to obtain c}$$

$$2 = \cos(0)\cdot(0 + c) \qquad | \text{ solve to c}$$

$$c = 2 \qquad | \text{ state special solution}$$

$$y = \cos(x)\cdot(x + 2)$$

Exercises on differential equations Nos 7 and 8

5.6.4. Slope fields and the Euler method

5.6.4.1. Slope fields

The differential equation $\frac{dy}{dx} = f(x, y)$ gives the gradient or slope $\frac{dy}{dx}$ of the function y(x) in any given point (x|y). The **slope field** is constructed by small arrows with gradient f(x,y) on each point (x|y). To save time and effort lines of constant gradient can be used to replace the arrows. The equations such an **isocline** with gradient c has the equation c = f(x,y).

Example

a) Construct the slope field of $\frac{dy}{dx} = x - y^2 + 2$ for all points with integer coordinates in the domain $-3 \le x,y \le 3$

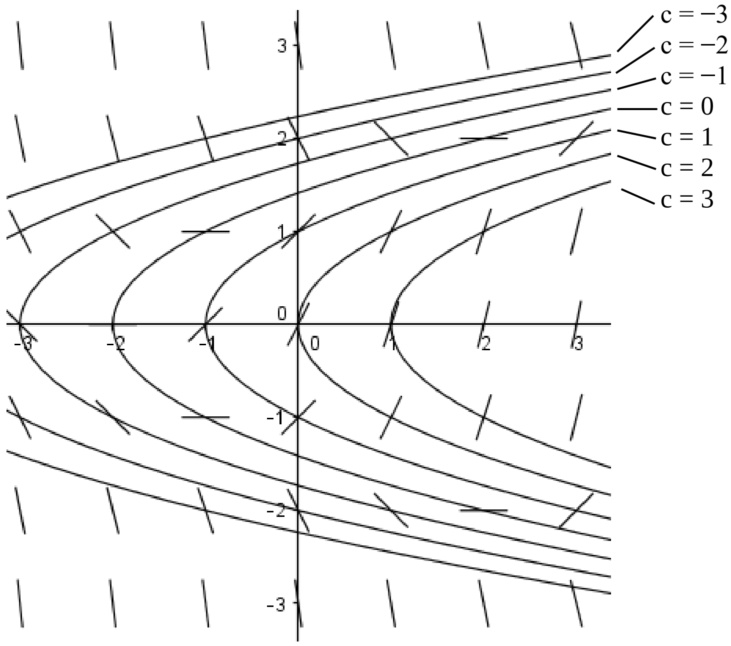
b) Draw the isoclines of $\frac{dy}{dx} = x - y^2 + 2$ in the domain $-3 \le x,y \le 3$ for slopes $c \in \{-3, -2, \dots, 3\}$

Solution
We use a table to plug in all 49 combinations of integer coordinates in the given domain.

For example the point (2|3) has the slope
$\frac{dy}{dx} = 2 - 3^2 + 2 = -5$

y \ x	−3	−2	−1	0	1	2	3
−3	−10	−9	−8	−7	−6	−5	−4
−2	−5	−4	−3	−2	−1	0	1
−1	−2	−8	0	1	2	3	4
0	−1	0	1	2	3	4	5
1	−2	−1	0	1	2	3	4
2	−5	−4	−3	−2	−1	0	1
3	−10	−9	−8	−7	−6	−5	−4

The isoclines $c = x - y^2 + 2 \Leftrightarrow y = \pm\sqrt{x+2-c}$ can be obtained by translation of $y = \sqrt{x}$ in direction of the x-axis c – 2 steps to the right.



5.6.4.2. The Euler method

To find an **approximate solution** for the differential equation $\frac{dy}{dx} = f(x, y)$ with given **initial value** $y(x_0) = y_0$ and **step length** Δx proceed as follows:

Start from the given point $(x_0|y_0)$, calculate the **gradient** $m_0 = f(x_0, y_0)$ and add the corresponding values Δx and $\Delta y = m_0 \cdot \Delta x$ to obtain the next point $(x_1|y_1) = (x_0 + \Delta x|y_0 + \Delta y)$.

Using the new point we repeat the above described steps to get a point sequence $(x_i|y_i)_{i \geq 0}$ which approximates the ideal solution y(x).

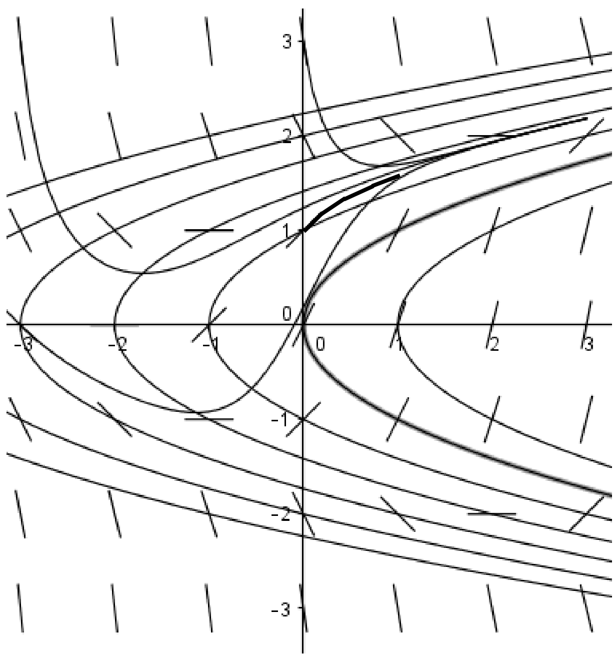
Example:
Using Euler's method with step length of 0,25, approximate the solution, y, of the differential equation $\frac{dy}{dx} = x - y^2 + 2$ when x = 0, given that y = 1 when x = 0.

Solution
It is convenient to organize the results in a table:

x	y	m
0,00	1,000	1
0,25	1,250	0,6875
0,50	1,422	0,4783
0,75	1,541	0,3740
1,00	1,6349	

⇒ Result $y(1) \approx 1{,}63$

Exercises on differential equations Nos9 and 10



5.6.5. Using Taylor polynomials

Basically the differential equation $\frac{dy}{dx} = f(x, y)$ gives a formula for the derivative of the function y. Hence it can be used repeatedly to develop the Taylor series $y(x) = y(x_0) + y'(x_0)\cdot(x - x_0) + \frac{y''(x_0)}{2!}\cdot(x - x_0)^2 + \ldots$. The coefficients are obtained as follows: $y(x_0) = y_0$; $y'(x_0) = f(x_0,y_0)$, $y''(x_0) = f'(x_0,y_0)$, …

Example

Find a Taylor polynomial of degree 3 to approximate the solution of the differential equation $\frac{dy}{dx} = x^2 + y^2$ with y(1) = 0 and use this polynomial to find an approximation of y(1,1) to 4 decimal points.

Solution

By implicit differentiation we obtain $y''(x) = (x^2 + y^2)' = 2x + 2yy'$ and $y'''(x) = (2x + 2yy')' = 2 + 2y'^2 + 2yy''$.
We plug in $y(1) = 0$; $y'(1) = 1^2 + 0^2 = 1$, $y''(1) = 2\cdot1 + 2\cdot0\cdot1 = 2$ and $y'''(1) = 2 + 2\cdot1^2 + 2\cdot0\cdot2^2 = 4$.

The Taylor polynomial is $y(x) \approx y(1) + y'(1)\cdot(x - 1) + \frac{y''(1)}{2!}\cdot(x - 1)^2 + \frac{y'''(1)}{3!}\cdot(x - 1)^3 = (x - 1) + (x - 1)^2 + \frac{2}{3}(x - 1)^3$.

The approximate value at x = 1,1 is y(1,1) 0 $0,1 + 0,01 + \frac{2}{3}\cdot0,001 \approx 0,1107$.

Exercises on differential equations No 11