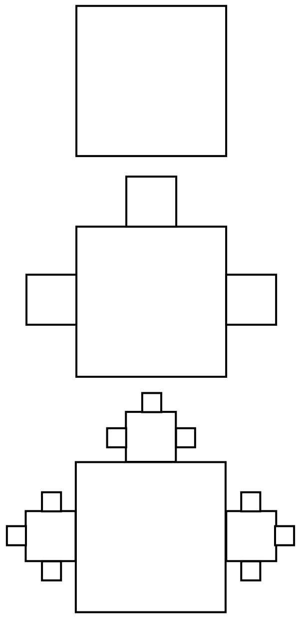
5.7. Exercises on sequences and series



Exercise 1: linear and exponential growth

A square with side length 1 dm consecutively grows smaller squares on three of its four sides. The squares added in the n th step have only **one third** of the side length of the squares added in the previous step n – 1

a) Calculate the circumference U_n after n = 0, 1, 2, 3 and 4 steps.
b) What is the difference $U_{n+1} - U_n$ of the circumference from step n to step n + 1?
c) Find an expression of U_{n+1} in terms of U_n.
d) Find an expression of U_n in terms of n.
e) Calculate the area A_n of the shape after n = 1, 2, 3 and 4 steps
f) What is the difference $A_{n+1} - A_n$ of the area from step n to step n + 1?
g) Find an expression of A_{n+1} in terms of A_n.
h) Find an expression of A_n in terms of n.

Hint: $1 + x + x^2 + x^3 + ... + x^n = \frac{1-x^{n+1}}{1-x}$.

i) Calculate A_{100} and U_{100}. Describe the trend of U_n and A_n for ever larger n. Draw conclusions about the circumference of natural places like for example an island.
j) Determine the limit $\lim\limits_{n\to\infty} A_n$.

Exercise 2: Calculating terms with recursive and general formulae

Calculate the first 5 terms $a_0, ..., a_4$ and draw a diagram. Can you guess the limit for $n \to \infty$?

a) $a_n = 100\cdot 2^{-n}$
b) $a_n = 100 - 50\cdot 2^{-n}$
c) $a_n = \frac{1}{n+1}$
d) $a_n = (n + 1)(n + 2)$
e) $a_{n+1} = a_n + \frac{1}{2}$ with $a_0 = 3$
f) $a_{n+1} = a_n + \frac{1}{2} a_n$ with $a_0 = 3$
g) $a_{n+1} = a_n + \frac{1}{2}(5 - a_n)$ with $a_0 = 3$
h) $a_{n+1} = a_n + \frac{1}{20} a_n\cdot(5 - a_n)$ with $a_0 = 3$

Exercise 3: Finding recursive and general formulae for given terms of a sequence

Find the recursive formula and the general formula for the sequence with the terms given:

a) $a_0 = 1; a_1 = 3; a_2 = 5; a_3 = 7; a_4 = 9$
b) $a_0 = 3; a_1 = 6; a_2 = 12; a_3 = 24; a_4 = 48$
c) $a_0 = 2; a_1 = 6; a_2 = 18; a_3 = 54; a_4 = 162$
d) $a_0 = 2; a_1 = 5; a_2 = 10; a_3 = 17; a_4 = 26$
e) $a_0 = 0; a_1 = \frac{1}{2}; a_2 = \frac{2}{3}; a_3 = \frac{3}{4}; a_4 = \frac{4}{5}$
f) $a_0 = 1; a_1 = \frac{2}{3}; a_2 = \frac{4}{9}; a_3 = \frac{8}{27}; a_4 = \frac{16}{81}$
g) $a_0 = -1; a_1 = 1; a_2 = \frac{7}{5}; a_3 = \frac{11}{7}; a_4 = \frac{5}{3}$
h) $a_0 = 0; a_1 = \frac{1}{3}; a_2 = \frac{4}{9}; a_3 = \frac{1}{3}; a_4 = \frac{16}{81}$

Exercise 4: Converting general formulae into recursive formulae

Find a recursive formula for the given general formula

a) $a_n = 3n + 2$
b) $a_n = n^2 - 2n$
c) $a_n = 3^{-n}$
d) $a_n = \frac{n}{n+1}$

Exercise 5: Converting recursive formulae into general formulae

Find an general formula for the given recursive formula

a) $a_{n+1} = a_n - 3$ with $a_0 = 2$
b) $a_{n+1} = 0{,}8\cdot a_n$ with $a_0 = 20$
c) $a_{n+1} = a_n + 2n + 2$ with $a_0 = 0$
d) $a_{n+1} = a_n + 2n + 1$ with $a_0 = 0$

Exercise 6: Monotonicity of a sequence
Examine the following sequences on monotonic increasing or decreasing behavior:

a) $a_n = \frac{n-1}{n+1}$ b) $a_n = \sqrt{n^2 - n}$ c) $a_n = n^3 - 3n^2$ d) $a_n = n^2 \cdot 2^{-n}$

Exercise 7: Boundedness of a sequence
Examine the following sequences on upper and lower bounds:

a) $a_n = \frac{n}{n+1}$ b) $a_n = \sqrt{n^2 + n}$ c) $a_n = n^2 - n^3$ d) $a_n = n^2 \cdot 3^{-n}$

Exercise 8: Limit of a sequence
Find the limit $\lim\limits_{n\to\infty} a_n$ and give reasons.

a) $a_n = \frac{n+2}{n+1}$ b) $a_n = \frac{1}{n}\sqrt{n^2 + n}$ c) $a_n = \frac{1}{n}\sin(n)$ d) $a_n = n^3 \cdot 2^{-n}$

Exercise 9: Convergence of a sequence
Examine the sequence (a_n) for $n \geq 1$ on monotonicity and boundedness. Then draw a conclusion about its convergence:

a) $a_n = \frac{1-4n}{1+2n}$ b) $a_n = \frac{n-1}{2n}$ c) $a_n = \frac{2^n + 3^n}{2^n - 3^n}$ d) $a_n = \frac{n\sqrt{n}+10}{n^2}$

Exercise 10: Series and sigma notation
Fill in the blanks:

generating sequence a_n	series $\sum\limits_{k=0}^{n} a_k$	corresponding function f(x)	Integral $\int_1^{n+1} f(x)dx$
$\frac{1}{n}$			
	$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$		
	$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$		
		$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$	

Exercise 11: Arithmetic series

a) Calculate $\sum\limits_{k=0}^{20} a_k$ and $\sum\limits_{k=60}^{100} a_k$ for the sequence $a_n = 2 + \frac{n}{10}$.

b) Calculate $\sum\limits_{k=0}^{16} a_k$ and $\sum\limits_{k=40}^{80} a_k$ for the sequence (a_n) with $a_0 = 3$ and $a_{n+1} = a_n + \frac{1}{2}$

c) Calculate the initial value a_0 and the common difference d for the arithmetic sequence (a_n) with $\sum\limits_{k=0}^{10} a_k = 22$ and $\sum\limits_{k=0}^{6} a_k = 7$.

d) Calculate the common difference d for the arithmetic sequence (a_n) with $\sum\limits_{k=10}^{90} a_k = 31$ and initial value $a_0 = 1$.

Exercise 12: Geometric series

a) Calculate $\sum\limits_{k=0}^{20} a_k$ and $\sum\limits_{k=70}^{100} a_k$ for the sequence $a_n = 100 \cdot 0{,}9^n$.

b) Calculate $\sum\limits_{k=0}^{10} a_k$ and $\sum\limits_{k=40}^{50} a_k$ for the sequence (a_n) with $a_0 = 3$ and $a_{n+1} = 1{,}2 \cdot a_n$.

c) Calculate initial value a_0 and common ratio q for the geometric sequence (a_n) with $\sum\limits_{k=0}^{7} a_k = 641$ and $\sum\limits_{k=0}^{3} a_k = 625$.

d) Calculate initial value a_0 and common ratio q for the geometric sequence (a_n) with $\sum\limits_{k=0}^{4} a_k = 336{,}16$ and limit $\sum\limits_{k=0}^{\infty} a_k = 500$

Exercise 13: Limit of a series

Examine the series $\sum_{k=1}^{n} a_k$ on monotonicity and boundedness. Then draw a conclusion about its convergence:

a) $\sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}$

b) $\sum_{k=0}^{n} \frac{1}{(2k+1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... + \frac{1}{(2n+1)^2}$

c) $\sum_{k=1}^{n} \frac{1}{3^k} = \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + ... + \frac{1}{3^n}$

d) $\sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n}$

Exercise 14: Mathematical induction

Prove with mathematical induction

a) $2 + 4 + 6 + ... + 2n = n(n + 1)$ for $n \geq 1$

b) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6} n(n + 1)(2n + 1)$ for $n \geq 1$

c) $6 + 24 + 60 + 120 + \dots + n(n + 1)(n + 2) = \frac{1}{4} n(n + 1)(n + 2)(n + 3)$

d) $x^0 + x^1 + x^2 + x^3 + ... + x^n = \frac{1-x^{n+1}}{1-x}$ for $n \geq 0$.

e) $1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$ for $n \geq 1$

f) The sequence a_n with $a_1 = 2$ and $a_{n+1} = a_n + (n + 1)(n + 2)$ has the general formula $a_n = \frac{1}{3} n(n + 1)(n + 2)$.

g) The sequence a_n mit $a_1 = \frac{3}{4}$ and $a_{n+1} = a_n - \frac{a_n}{(3n+4)(2n+1)}$ has the general formula $a_n = \frac{2n+1}{3n+1}$.

h) $8^n - 1$ is divisible by 7 for $n \geq 1$

i) $n^3 - n$ is divisible by 6 for $n \geq 2$

j) $(1 + x)^n > 1 + nx$ for $n \geq 2$, $x > -1$ and $x \neq 0$ (**Bernoulli's inequality**)

Exercise 15: Monotonicity and Boundedness

a) Show by mathematical induction that the sequence $(a_n)$with $a_0 = 3$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$ is positive for all $n \in \mathbb{N}$

b) Solve the inequality $\frac{1}{2}\left(x + \frac{3}{x}\right) > \sqrt{3}$ to x and show that the sequence from a) has the lower bound $\sqrt{3}$.

c) Show that the sequence from a) is monotonically decreasing.

d) Find the limit $a = \lim_{n\to\infty} a_n$ of the sequence from a).

Hint: For $n \to \infty$ holds $a_n = a_{n+1} = \lim_{n\to\infty} a_n = a$. Thus in the recursive formula you can plug in $a_n = a_{n+1} = a$ and the solve to a.

5.7. Solutions to the exercises on sequences and series

Exercise 1: Linear and exponential growth

All lengths are given in dm, all areas in dm^2:

a) $U_0 = 4$, $U_1 = 6$, $U_2 = 8$, $U_3 = 10$ and $U_4 = 12$

b) $U_{n+1} - U_n = 2$ (**linear growth**

c) $U_{n+1} = U_n + 2$ (**recursive formula**)

d) $U_n = 4 + 2n$ (**general formula**)

e) $A_0 = 1$, $A_1 = 1 + \frac{1}{3} \approx 1{,}33$, $A_2 = 1 + \frac{1}{3} + \frac{1}{9} \approx 1{,}44$, $A_3 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \approx 1{,}48$

and $A_4 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \approx 1{,}49$

f) $A_{n+1} - A_n = 3^{n+1} \cdot \left(\frac{1}{3^{n+1}}\right)^2 = \left(\frac{1}{3}\right)^{n+1}$ (**exponential growth**)

g) $A_{n+1} = A_n + \left(\frac{1}{3}\right)^{n+1}$ (**recursive formula**)

h) $A_n = 1 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + ... + \left(\frac{1}{3}\right)^n = \frac{1-1/3^{n+1}}{1-1/3} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right) = \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n$.(**general formula**)

i) $U_{100} = 204$ and $A_{100} \approx 1{,}5 \Rightarrow$ The circumference grows beyond all bounds but the area is bounded!

j) $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right) = \frac{3}{2}$

Exercise 2: Calculating terms with recursive and general formulae

a) $a_0 = 100$; $a_1 = 50$; $a_2 = 25$; $a_3 = 12{,}5$; $a_4 = 6{,}25$ with $\lim\limits_{n\to\infty} a_n = 0$

b) $a_0 = 50$; $a_1 = 75$; $a_2 = 87{,}5$; $a_3 = 93{,}75$; $a_4 = 96{,}875$ with $\lim\limits_{n\to\infty} a_n = 100$

c) $a_0 = 1$; $a_1 = \frac{1}{2}$; $a_2 = \frac{1}{3}$; $a_3 = \frac{1}{4}$; $a_4 = \frac{1}{5}$ with $\lim\limits_{n\to\infty} a_n = 0$

d) $a_0 = 2$; $a_1 = 6$; $a_2 = 12$; $a_3 = 20$; $a_4 = 30$ and no limit

e) $a_0 = 3$; $a_1 = 3{,}5$; $a_2 = 4$; $a_3 = 4{,}5$; $a_4 = 5$ and no limit

f) $a_0 = 3$; $a_1 = 4{,}5$; $a_2 = 6{,}75$; $a_3 = 10{,}125$; $a_4 = 15{,}1875$ and no limit

g) $a_0 = 3$; $a_1 = 4$; $a_2 = 4{,}5$; $a_3 = 4{,}75$; $a_4 = 4{,}875$ with $\lim\limits_{n\to\infty} a_n = 5$

h) $a_0 = 3$; $a_1 = 3{,}3$; $a_2 \approx 3{,}74$; $a_3 \approx 3{,}98$; $a_4 = 4{,}18$ with $\lim\limits_{n\to\infty} a_n = 5$

Exercise 3: Finding general and recursive formulae from given terms of a sequence

a) $a_{n+1} = a_n + 2$ with $a_0 = 1 \Rightarrow a_n = 1 + 2n$

b) $a_{n+1} = 2a_n$ with $a_0 = 3 \Rightarrow a_n = 3 \cdot 2^n$

c) $a_{n+1} = 3a_n$ with $a_0 = 2 \Rightarrow a_n = 2 \cdot 3^n$

d) $a_{n+1} = a_n + 2n + 3$ with $a_0 = 2 \Rightarrow a_n = n^2 + 1$

e) $a_{n+1} = a_n + \frac{1}{(n+1)(n+2)}$ with $a_0 = 0 \Rightarrow a_n = \frac{n}{n+1}$

f) $a_{n+1} = \frac{2}{3} a_n$ with $a_0 = 1 \Rightarrow a_n = \left(\frac{2}{3}\right)^n$

g) $a_{n+1} = a_n + \frac{6}{(2n+3)(2n+1)}$ with $a_0 = -1 \Rightarrow a_n = \frac{4n-1}{2n+1}$

h) $a_{n+1} = \frac{1}{3}\left(\frac{n+1}{n}\right)^2 a_n$ for $n \geq 1$ with $a_1 = 1$ and $a_0 = 0 \Rightarrow a_n = \frac{n^2}{3^n}$

Exercise 4: Converting general formulae into general formulae

a) $a_{n+1} = a_n + 3$ with $a_0 = 2$

b) $a_{n+1} = a_n + 2n - 1$ with $a_0 = 0$

c) $a_{n+1} = \frac{1}{3} a_n$ with $a_0 = 1$

d) $a_{n+1} = a_n + \frac{1}{(n+1)(n+2)}$ with $a_0 = 0$

Exercise 5: Converting recursive formulae into general formulae

a) $a_n = -3n + 2$ b) $a_n = 20 \cdot 0{,}8^n$ c) $a_n = n(n + 1)$ d) $a_n = n^2$

Exercise 6: Monotonicity of a sequence

a) monotonically increasing, since $\frac{a_{n+1}}{a_n} = \frac{n(n+1)}{(n+2)(n-1)} = \frac{n^2+n}{n^2+n-2} > 1$ for all $n \in \mathbb{N}$.

b) monotonically increasing, since $\frac{a_{n+1}}{a_n} = \frac{\sqrt{(n+1)^2-(n+1)}}{\sqrt{n^2-n}} = \sqrt{\frac{n^2+n}{n^2-n}} > 1$ for all $n \in \mathbb{N}$.

c) monotonically increasing for $n \geq 2$,

since $a_{n+1} - a_n = (n+1)^3 - 3(n+1)^2 - n^3 + 3n^2 = 3n^2 - 3n - 2 = 3(n^2 - n - \frac{2}{3}) > 0$ for $n \geq 2$

d) monotonically decreasing for $n \geq 2$,
since $a_{n+1} - a_n = (n+1)^2 \cdot 2^{-(n+1)} - n^2 \cdot 2^{-n} = 2^{-(n+1)}(n^2 + 2n + 1 - 2n^2) = 2^{-(n+1)}(-n^2 + 2n + 1) < 0$ for $n \geq 2$

Exercise 7: Boundedness of a sequence

a) Upper bound $S_o = 1$, since $a_n \leq 1 \Leftrightarrow \frac{n}{n+1} \leq 1$ for all $n \in \mathbb{N}$ and lower bound $S_u = 0$, since $a_n \geq 0 \Leftrightarrow \frac{n}{n+1} \geq 0$ for $n \in \mathbb{N}$

b) No upper bound S, since there is no $S \in \mathbb{R}$ with $a_n \leq S \Leftrightarrow \sqrt{n^2+n} \leq S \Leftrightarrow n^2 + n \leq S^2$ for all $n \in \mathbb{N}$.

Lower bound $S_u = 0$, since $a_n \geq 0 \Leftrightarrow \sqrt{n^2+n} \geq 0$ for all $n \in \mathbb{N}$.

c) Upper bound $S_o = 0$, since $a_n \leq 0 \Leftrightarrow n^2 - n^3 \leq 0 \Leftrightarrow n^2(1-n) \leq 0$ for all $n \in \mathbb{N}$.

No lower bound S, since there is no $S \in \mathbb{R}$ with $a_n \geq S \Leftrightarrow n^2 - n^3 \geq S$ for all $n \in \mathbb{N}$

d) Upper bound $S_o = 1$, since $a_n \leq 1 \Leftrightarrow n^2 \cdot 3^{-n} \leq 1 \Leftrightarrow n^2 \leq 3^n$ for all $n \in \mathbb{N}$.

Lower bound $S_u = 0$, since $n^2 \cdot 3^{-n} \geq 0$ for all $n \in \mathbb{N}$

Exercise 8: Limit of a sequence

We have to show that for any $\varepsilon > 0$ there is a n_e so that the condition $|a_\varepsilon - a| < \varepsilon$ holds for all $n > n_\varepsilon$.

a) $\lim_{n\to\infty} a_n = 1$, since $|a - a_n| \leq \varepsilon \Leftrightarrow |1 - \frac{n+2}{n+1}| \leq \varepsilon \Leftrightarrow \frac{1}{n+1} \leq \varepsilon \Leftrightarrow \frac{1}{\varepsilon} - 1 \leq n$ holds for all $n \geq n_\varepsilon = \frac{1}{\varepsilon} - 1$.

b) $\lim_{n\to\infty} a_n = 1$, since $|a - a_n| \leq \varepsilon \Leftrightarrow |1 - \frac{1}{n}\sqrt{n^2+n}| \leq \varepsilon \Leftrightarrow 1 - \sqrt{1+\frac{1}{n}} | \leq \varepsilon \Leftrightarrow \sqrt{1+\frac{1}{n}} \leq 1 - \varepsilon$

$\Leftrightarrow 1 + \frac{1}{n} \leq (1-\varepsilon)^2 \Leftrightarrow \frac{1}{1-(1-\varepsilon)^2} \leq n \Leftrightarrow \frac{1}{\varepsilon(2-\varepsilon)} \leq n$ holds for all $n \geq n_\varepsilon = \frac{1}{\varepsilon(2-\varepsilon)}$.

c) $\lim_{n\to\infty} a_n = 0$, since $|a - a_n| \leq \varepsilon \Leftrightarrow |\frac{1}{n}\sin(n)| \leq \varepsilon \Leftrightarrow \frac{1}{\varepsilon} \leq n$ holds for all $n \geq n_\varepsilon = \frac{1}{\varepsilon}$.

d) $\lim_{n\to\infty} a_n = 0$, since $|a - a_n| \leq \varepsilon \Leftrightarrow |n^3 \cdot 2^{-n}| \leq \varepsilon \Leftrightarrow \frac{1}{\varepsilon} \leq \frac{2^n}{n^3}$ holds for all $n \geq 10$

Exercise 9: Convergence of a sequence

a) Boundedness: $a_n = \frac{1-4n}{1+2n} = -2 + \frac{3}{1+2n} \Rightarrow -2 < a_n < 1$, since $a_n + 2 = \frac{3}{1+2n}$ and $0 < \frac{3}{1+2n} < 3$

Monotonicity: $a_{n+1} - a_n = \frac{3}{1+2(n+1)} - \frac{3}{1+2n} = \frac{3}{2n+3} - \frac{3}{2n+1} = < 0 \Rightarrow (a_n)$ decreases monotonically (to $\lim_{n\to\infty} a_n = -2$)

b) Boundedness: $a_n = \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n} \Rightarrow 0 < a_n < \frac{1}{2}$, since $0 < \frac{1}{2n} < \frac{1}{2}$.

Monotonicity: $a_{n+1} - a_n = -\frac{1}{2(n+1)} + \frac{1}{2n} > 0 \Rightarrow (a_n)$ increases monotonically (to $\lim_{n\to\infty} a_n = \frac{1}{2}$)

c) Boundedness: $a_n = \frac{2^n+3^n}{2^n-3^n} = \frac{3^n\left(1+\left(\frac{2}{3}\right)^n\right)}{-3^n\left(1-\left(\frac{2}{3}\right)^n\right)} = -\frac{1+\left(\frac{2}{3}\right)^n}{1-\left(\frac{2}{3}\right)^n} \Rightarrow -2 < a_n < 0$, since $0 < \frac{1+\left(\frac{2}{3}\right)^n}{1-\left(\frac{2}{3}\right)^n} < 1 + \left(\frac{2}{3}\right)^n < 2$

Monotonicity: $a_{n+1} - a_n = \frac{2^n+3^n}{2^n-3^n} - \frac{2^{n+1}+3^{n+1}}{2^{n+1}-3^{n+1}} = \frac{(2^{n+1}-3^{n+1})(2^n+3^n)-(2^{n+1}+3^{n+1})(2^n-3^n)}{(2^{n+1}-3^{n+1})(2^n-3^n)}$

$= \frac{2^{2n+1}-6^n+3^{2n+1}-(2^{2n+1}+6^n-3^{2n+1})}{(2^{n+1}-3^{n+1})(2^n-3^n)} = \frac{2\cdot 3^{2n+1}}{(2^{n+1}-3^{n+1})(2^n-3^n)} > 0$

$\Rightarrow$ (a_n) increases monotonically (to $\lim_{n\to\infty} a_n = -1$)

d) Boundedness: $a_n = \frac{n\sqrt{n}+10}{n^2} = \frac{1}{\sqrt{n}} + \frac{10}{n^2} \Rightarrow 0 < a_n < 1 + 10 = 11$

Monotonicity: $a_{n+1} - a_n = \frac{1}{\sqrt{n+1}} + \frac{10}{(n+1)^2} - \frac{1}{\sqrt{n}} - \frac{10}{n^2} = \left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{10}{(n+1)^2} - \frac{10}{n^2}\right) < 0$, since both brackets < 0

$\Rightarrow$ (a_n) decreases monotonically (to $\lim_{n\to\infty} a_n = 0$)

Exercise 10: Series and sigma notation

Generating sequence a_n	Series $\sum_{k=0}^{n} a_k$	Corresponding function f(x)	Integral $\int_1^{n+1} f(x)dx$
$\frac{1}{n}$	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$	$\frac{1}{x}$	ln(n + 1)
$\frac{1}{3^n}$	$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$	$\frac{1}{3^x}$	$3\cdot\ln(3)\cdot\left(1-\frac{1}{3^n}\right)$
$\frac{1}{n^2}$	$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$	$\frac{1}{x^2}$	$1 - \frac{1}{n+1}$
$\frac{1}{n(n+1)}$	$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots$	$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$	$\ln(2) + \ln\left(\frac{n+1}{n+2}\right)$

Exercise 11: Arithmetic series

a) $\sum_{k=0}^{20} a_k = 63$ and $\sum_{k=60}^{100} a_k = \sum_{k=0}^{100} a_k - \sum_{k=0}^{59} a_k = 707 - 297 = 410$.

b) $\sum_{k=0}^{16} a_k = 119$ and $\sum_{k=40}^{80} a_k = \sum_{k=0}^{80} a_k - \sum_{k=0}^{39} a_k = 1863 - 510 = 1353$.

c) $\sum_{k=0}^{10} a_k = 11a_0 + 55d = 22 \Leftrightarrow a_0 + 5d = 2$ and $\sum_{k=0}^{6} a_k = 7a_0 + 21d = 7 \Leftrightarrow a_0 + 3d = 1$ result in $a_0 = -\frac{1}{2}$ and $d = \frac{1}{2}$

d) $\sum_{k=0}^{90} a_k = 91\cdot 1 + 4095d = \sum_{k=0}^{9} a_k + 31 = 10\cdot 1 + 45d + 31 \Leftrightarrow 91 + 4095d = 41 + 45d \Leftrightarrow 50 = 4050\,d$ result in $d = \frac{1}{81}$

Exercise 12: Geometric series

a) $\sum_{k=0}^{20} a_k = 1000\cdot(1 - 0{,}9^{21}) \approx 890{,}581$ and $\sum_{k=70}^{100} a_k = 1000\cdot(1 - 0{,}9^{101}) \approx 999{,}976$.

b) $\sum_{k=0}^{10} a_k = 15\cdot(1{,}2^{11} - 1) \approx 96{,}45$ and $\sum_{k=40}^{50} a_k = 15\cdot(1{,}2^{51} - 1) \approx 163\,792{,}89$

c) $\sum_{k=0}^{7} a_k = a_0\cdot\frac{q^8-1}{q-1} = 641$ and $\sum_{k=0}^{3} a_k = a_0\cdot\frac{q^4-1}{q-1} = 625$ result in $\frac{q^8-1}{q^4-1} = \frac{641}{625} \Leftrightarrow \frac{(q^4-1)(q^4+1)}{q^4-1} = 1{,}0256$

$\Leftrightarrow q^4 = 0{,}256 \Rightarrow$ common ratio $q = 0{,}4$ and initial value $a_0 \approx 384{,}85$

d) $\sum_{k=0}^{4} a_k = a_0\cdot\frac{q^5-1}{q-1} = 336{,}16$ and limit $\sum_{k=0}^{\infty} a_k = a_0\cdot\frac{1}{1-q} = 500$ result in $q^5 - 1 = -\frac{336{,}16}{500} \Leftrightarrow q^5 = 0{,}32768 \Leftrightarrow$ common ratio $q = 0{,}8$ and initial value $a_0 = 100$.

Exercise 13: Limit of a series

a) $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is monotonically increasing, since $s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0$ for all $n \in \mathbb{N}$. (s_n) is bounded above, since $s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} < 1 + \int_1^n \frac{1}{x^2}dx = 1 + \left[-\frac{1}{x}\right]_1^n = 2 - \frac{1}{n} < 2$ for all $n \in \mathbb{N}$. Therefore (s_n) converges to a $\lim_{n\to\infty} s_n \leq 2$. L. Euler showed in 1736, that $\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1{,}64$

b) $s_n = \sum_{k=0}^{n} \frac{1}{(2k+1)^2}$ is monotonically increasing, since $s_{n+1} - s_n = \frac{1}{(2n+3)^2} > 0$ for all $n \in \mathbb{N}$. (s_n) is bounded above, since $s_n = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... + \frac{1}{(2n+1)^2} < 1 + \int_0^n \frac{1}{(2x+1)^2}dx = 1 + \left[-\frac{1}{2(2x+1)}\right]_0^n = \frac{3}{2} - \frac{1}{4n+2} < \frac{3}{2}$ for all $n \in \mathbb{N}$.

Therefore (s_n) converges to $\lim_{n\to\infty} s_n \leq \frac{3}{2}$. L. Euler showed, that $\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \approx 1{,}23$

c) $s_n = \sum_{k=0}^{n} \frac{1}{3^k}$ is monotonically increasing, since $s_{n+1} - s_n = \frac{1}{3^n} > 0$ for all $n \in \mathbb{N}$. (s_n) is bounded above, since $s_n = 1 + \frac{1}{3^1} + \frac{1}{3^2} + ... + \frac{1}{3^n} < \int_{-1}^n \frac{1}{3^x}dx = \int_{-1}^n e^{-\ln(3)\cdot x}dx = \left[-\frac{1}{\ln(3)}\cdot e^{-\ln(3)\cdot x}\right]_{-1}^n = \frac{3}{\ln(3)} - \frac{1}{\ln(3)\cdot 3^n} < \frac{3}{\ln(3)}$ for $n \in \mathbb{N}$.

Therefore (s_n) converges to $\lim_{n\to\infty} s_n \leq \frac{3}{\ln(3)}$. The exact value of this limit can be obtained by using the summation rule for geometric series: $\sum_{k=0}^{\infty} \frac{1}{3^k} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{3^k} = \lim_{n\to\infty} 1\cdot\frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.

d) $s_n = \sum_{k=1}^{n} \frac{1}{n+k}$ is monotonically increasing, since $s_{n+1} - s_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$ for $n \in \mathbb{N}$.

(s_n) is bounded above, since $s_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n} < \int_0^n \frac{1}{n+x}dx = \ln(n+x)\,\Big|_0^n = \ln(2n) - \ln(n) = \ln(2)$ for all $n \geq 1$ and $s_0 = 1$. Therefore (s_n) converges to $\lim_{n\to\infty} s_n \leq \ln(2)$.

Exercise 14: Mathematical induction

a) **Base case** n = 1: $2 = \cdot 1 \cdot (1 + 1)$

Inductive step $n \Rightarrow n + 1$:
Inductive hypothesis for some n: $2 + 4 + 6 + ... + 2n = n(n + 1)$
We have to show the statement for n + 1: $2 + 4 + 6 + ... + 2n + 2(n + 2 = (n + 1)(n + 2)$
Plug the hypothesis into the left side:
$2 + 4 + 6 + ... + 2n + 2(n + 2) = n(n + 1) + 2n + 2 = n^2 + 3n + 2$
Right side: $(n + 1)(n + 2) = n^2 + 3n + 2$ **qed**

b) **Base case** n = 1: $1^2 = \frac{1}{6} 1(1 + 1)(2 + 1) = 1$

Inductive step $n \Rightarrow n + 1$:

Inductive hypothesis for some n: $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6} n(n + 1)(2n + 1)$

We have to show the statement for n + 1: $1^2 + 2^2 + 3^2 + ... + n^2 + (n + 1)^2 = \frac{1}{6} (n + 1)(n + 2)(2n + 3)$

Plug the hypothesis into the left side:

$1^2 + 2^2 + 3^2 + ... + n^2 + (n + 1)^2 = \frac{1}{6} n(n + 1)(2n + 1) + (n + 1)^2 = \frac{1}{3} n^3 + \frac{3}{2} n^2 + \frac{13}{6} n + 1$

Right side: $\frac{1}{6} (n + 1)(n + 2)(2n + 3) = \frac{1}{6} (2n^3 + 9n^2 + 13n + 6) = \frac{1}{3} n^3 + \frac{3}{2} n^2 + \frac{13}{6} n + 1 =$ **left side**, **qed**

c) **Base case** n = 1: $6 = \frac{1}{4} 1\cdot 2\cdot 3\cdot 4 = 6$

Inductive step n ⇒ n + 1:

Inductive hypothesis for some n: $6 + 24 + 60 + 120 + \ldots + n(n + 1)(n + 2) = \frac{1}{4} n(n + 1)(n + 2)(n + 3)$

We have to show the statement for n + 1:

$$6 + 24 + 60 + 120 + \ldots + n(n + 1)(n + 2) + (n + 1)(n + 2)(n + 3) = \frac{1}{4}(n + 1)(n + 2)(n + 3)(n + 4)$$

Plug the hypothesis into the left side:

$$6 + 24 + 60 + 120 + \ldots + n(n + 1)(n + 2) + (n + 1)(n + 2)(n + 3) = \frac{1}{4} n(n + 1)(n + 2)(n + 3) + (n + 1)(n + 2)(n + 3)$$

$= \frac{1}{4}(n + 1)(n + 2)(n + 3)(n + 4) =$ **right side**, **qed**.

d) **Base case** n = 0: $1 = \frac{1-x^1}{1-x} = 1$

Inductive step n ⇒ n + 1:

Inductive hypothesis for some n: $x^0 + x^1 + x^2 + x^3 + ... + x^n = \frac{1-x^{n+1}}{1-x}$

We have to show the statement for n + 1: $x^0 + x^1 + x^2 + x^3 + ... + x^n + x^{n+1} = \frac{1-x^{n+2}}{1-x}$

Plug the hypothesis into the left side:

$x^0 + x^1 + x^2 + x^3 + ... + x^n + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1} = \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} = \frac{1-x^{n+2}}{1-x}$ = **right side, qed**

e) **Base case** n = 1: $1 = \frac{1-2x+x^2}{(1-x)^2} = 1$

Inductive step n ⇒ n + 1:

Inductive hypothesis for some n: $1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$

We have to show the statement for n + 1: $1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} + (n + 1)x^n = \frac{1-(n+2)x^{n+1}+(n+1)x^{n+2}}{(1-x)^2}$

Plug the hypothesis into the left side:

$$\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} + (n + 1)x^n = \frac{1-(n+1)x^n+nx^{n+1}+(1-x)^2(n+1)x^n}{(1-x)^2}$$

$$= \frac{1-(n+1)x^n+nx^{n+1}+(n+1)x^n-2(n+1)x^{n+1}+(n+1)x^{n+2}}{(1-x)^2}$$

$= \frac{1-(n+2)x^{n+1}+(n+1)x^{n+2}}{(1-x)^2}$ = **right side, qed**

f) **Base case** n = 1: $2 = \frac{1}{3}\cdot 1\cdot 2\cdot 3 = 2$

Inductive step n ⇒ n + 1:

Inductive hypothesis for some n: $a_n = \frac{1}{3} n(n + 1)(n + 2)$

We have to show the statement for n + 1: $a_{n+1} = \frac{1}{3}(n + 1)(n + 2)(n + 3)$

Plug the hypothesis into the left side:

$:a_{n+1} = a_n + (n+ 1)(n + 2) = \frac{1}{3} n(n + 1)(n + 2) + (n + 1)(n + 2) = \frac{1}{3} n^3 + n^2 + \frac{2}{3} n + n^2 + 3n + 2 = \frac{1}{3} n^3 + 2n^2 + \frac{11}{3} n + 2.$

Right side: $\frac{1}{3}(n + 1)(n + 2)(n + 3) = \frac{1}{3}(n^2 + 3n + 2)(n + 3) = \frac{1}{3} n^3 + 2n^2 + \frac{11}{3} n + 2 =$ **left side**, **qed.**

g) **Base case for n = 1**: $\frac{3}{4} = \frac{2+1}{3+1} = \frac{3}{4}$

Inductive step n ⇒ n + 1:

Inductive hypothesis for some n: $a_n = \frac{2n+1}{3n+1}$

We have to show the statement for n + 1: $a_{n+1} = \frac{2n+3}{3n+4}$

Plug the hypothesis into the left side:

$$a_{n+1} = a_n - \frac{a_n}{(3n+4)(2n+1)} = \frac{2n+1}{3n+1}\cdot\left(1-\frac{1}{(3n+4)(2n+1)}\right) = \frac{2n+1}{3n+1}\cdot\frac{6n^2+11n+3}{(3n+4)(2n+1)} = \frac{2n+1}{3n+1}\cdot\frac{(3n+1)(2n+3)}{(3n+4)(2n+1)} =$$

$\frac{2n+3}{3n+4}$ = **right side**, **qed.**

h) **Base case** n = 1: $8^1 - 1 = 7$ obviously is divisible by 7

Inductive step n ⇒ n + 1:
Inductive hypothesis for some n: $8^n - 1$ is divisible by 7
We have to show the statement for n + 1: $8^{n+1} - 1 = 8\cdot 8^n - 1 = 7\cdot 8^n + 8^n - 1$ is divisible by 7
Obviously the first summand $7\cdot 8^n$ 1 is divisible by 7. **According to the inductive hypothesis** the second summand 8n − 1 too is divisible by 7 and therefore the complete sum is divisible by 7. **qed**

i) **Base case** n = 1: $2^3 - 2 = 6$ 1 is divisible by 6

Inductive step n ⇒ n + 1:
Inductive hypothesis for some n: $n^3 - n$ is divisible by 6
We have to show the statement for n + 1:

$(n + 1)^3 - (n + 1) = n^3 + 3n^2 + 2n = (n^3 - n) + (3n^2 + 3n) = (n^3 - n) + 6\cdot\frac{1}{2}n(n + 1)$ is divisible by 6

According to the inductive hypothesis the left summand $n^3 - n$ is divisible by 6. But since either n or n + 1 is even, $\frac{1}{2}n(n + 1)$ is an integer. Therefore the right summand $6\cdot\frac{1}{2}n(n + 1)$ too is divisible by 6 and so is the complete sum. **qed**

j) **Base case** n = 2: $(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$, since $x^2 > 0$·

Inductive step n ⇒ n + 1:
Inductive hypothesis for some n: $(1 + x)^n > 1 + nx$
We have to show the statement for n + 1: $(1 + x)^{n+1} > 1 + (n + 1)x$
Plug the hypothesis into the left side:

$(1 + x)^{n+1}$ $(1 + x)^n(1 + x) > (1 + nx)(1 + x)$, since we assume $1 + x > 0$
$= 1 + (n + 1)x + x^2$
$> 1 + (n + 1)x$, since $x^2 > 0$
= **right side, qed**

Aufgabe 15: Mathematical induction, Monotonicity and Boundedness

a) **Base case** n = 0: $a_0 = 3 > 0$·

Inductive step n ⇒ n + 1:
Inductive hypothesis for some n: $a_n > 0$
We have to show the statement for n + 1: $a_{n+1} > 0$

Use the hypothesis: $a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right) > 0$, since $a_n > 0$, qed.

b) $\frac{1}{2}\left(x + \frac{3}{x}\right) > \sqrt{3} \Leftrightarrow x + \frac{3}{x} > 2\sqrt{3} \overset{x>0 \text{ because of a)}}{\Leftrightarrow} x^2 + 3 > 2\sqrt{3}\,x \Leftrightarrow x^2 - 2\sqrt{3}\,x + 3 > 0 \Leftrightarrow (x - \sqrt{3}) > 0$ for $x \in \mathbb{R}$.

c) $a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right) - a_n = \frac{1}{2}\left(\frac{3}{a_n} - a_n\right) = \frac{1}{2}\left(\frac{3 - a_n^2}{a_n}\right) > 0$, because in b) we have shown $a_n^2 < 3 \Rightarrow (a_n)$ is monotonically decreasing.

d) For $n \to \infty$ we have $a_n = a_{n+1} = \lim_{n\to\infty} a_n = a$ and so $a = \frac{1}{2}\left(a + \frac{3}{a}\right) \Leftrightarrow \frac{1}{2}a = \frac{1}{2}\frac{3}{a} \Leftrightarrow a^2 = 3 \overset{a>0 \text{ because of a)}}{\Leftrightarrow} a = \lim_{n\to\infty} a_n = \sqrt{3}$ **.**