5.7. Exercises on sequences and series

Exercise 1: linear and exponential growth
A square with side length 1 dm consecutively grows smaller squares on three of its four sides. The squares added in the \( n \) th step have only one third of the side length of the squares added in the previous step \( n - 1 \).

a) Calculate the circumference \( U_n \) after \( n = 0, 1, 2, 3 \) and 4 steps.
b) What is the difference \( U_{n+1} - U_n \) of the circumference from step \( n \) to step \( n + 1 \)?
c) Find an expression of \( U_{n+1} \) in terms of \( U_n \).
d) Find an expression of \( U_n \) in terms of \( n \).
e) Calculate the area \( A_n \) of the shape after \( n = 1, 2, 3 \) and 4 steps.
f) What is the difference \( A_{n+1} - A_n \) of the area from step \( n \) to step \( n + 1 \)?
g) Find an expression of \( A_{n+1} \) in terms of \( A_n \).
h) Find an expression of \( A_n \) in terms of \( n \).

Hint: \( 1 + x + x^2 + x^3 + ... + x^n = \frac{1 - x^{n+1}}{1 - x} \).
i) Calculate \( A_{100} \) and \( U_{100} \). Describe the trend of \( U_n \) and \( A_n \) for ever larger \( n \). Draw conclusions about the circumference of natural places like for example an island.
j) Determine the limit \( \lim_{n \to \infty} A_n \).

Exercise 2: Calculating terms with recursive and general formulae
Calculate the first 5 terms \( a_0, ..., a_4 \) and draw a diagram. Can you guess the limit for \( n \to \infty \)?

a) \( a_n = 100 \cdot 2^n \) e) \( a_{n+1} = a_n + \frac{1}{2} \) with \( a_0 = 3 \)
b) \( a_n = 100 - 50 \cdot 2^n \) f) \( a_{n+1} = a_n + \frac{1}{2} a_n \) with \( a_0 = 3 \)
c) \( a_n = \frac{1}{n+1} \) g) \( a_{n+1} = a_n + \frac{1}{2} (5 - a_n) \) with \( a_0 = 3 \)
d) \( a_n = (n+1)(n+2) \) h) \( a_{n+1} = a_n + \frac{1}{20} a_n (5 - a_n) \) with \( a_0 = 3 \)

Exercise 3: Finding recursive and general formulae for given terms of a sequence
Find the recursive formula and the general formula for the sequence with the terms given:

a) \( a_0 = 1; a_1 = 3; a_2 = 5; a_3 = 7; a_4 = 9 \) e) \( a_0 = 0; a_1 = \frac{1}{2}; a_2 = \frac{2}{3}; a_3 = \frac{3}{4}; a_4 = \frac{4}{5} \)
b) \( a_0 = 3; a_1 = 6; a_2 = 12; a_3 = 24; a_4 = 48 \) f) \( a_0 = 1; a_1 = \frac{2}{3}; a_2 = \frac{4}{9}; a_3 = \frac{8}{27}; a_4 = \frac{16}{81} \)
c) \( a_0 = 2; a_1 = 6; a_2 = 18; a_3 = 54; a_4 = 162 \) g) \( a_0 = -1; a_1 = 1; a_2 = \frac{7}{5}; a_3 = \frac{11}{7}; a_4 = \frac{5}{3} \)
d) \( a_0 = 2; a_1 = 5; a_2 = 10; a_3 = 17; a_4 = 26 \) h) \( a_0 = 0; a_1 = \frac{1}{3}; a_2 = \frac{3}{9}; a_3 = \frac{1}{3}; a_4 = \frac{16}{81} \)

Exercise 4: Converting general formulae into recursive formulae
Find a recursive formula for the given general formula:

a) \( a_n = 3n + 2 \) b) \( a_n = n^2 - 2n \) c) \( a_n = 3^n \) d) \( a_n = \frac{n}{n+1} \)

Exercise 5: Converting recursive formulae into general formulae
Find an general formula for the given recursive formula:

a) \( a_{n+1} = a_n - 3 \) with \( a_0 = 2 \) c) \( a_{n+1} = a_n + 2n + 2 \) with \( a_0 = 0 \)
b) \( a_{n+1} = 0,8a_n \) with \( a_0 = 20 \) d) \( a_{n+1} = a_n + 2n + 1 \) with \( a_0 = 0 \)
Exercise 6: Monotonicity of a sequence
Examine the following sequences on monotonic increasing or decreasing behavior:

a) \( a_n = \frac{n-1}{n+1} \)

b) \( a_n = \sqrt{n^2 - n} \)

c) \( a_n = n^3 - 3n^2 \)

d) \( a_n = n^3 \cdot 2^{-n} \)

Exercise 7: Boundedness of a sequence
Examine the following sequences on upper and lower bounds:

a) \( a_n = \frac{n}{n+1} \)

b) \( a_n = \sqrt{n^2 + n} \)

c) \( a_n = n^2 - n^3 \)

d) \( a_n = n^3 \cdot 3^{-n} \)

Exercise 8: Limit of a sequence
Find the limit \( \lim_{n \to \infty} a_n \) and give reasons.

a) \( a_n = \frac{n+2}{n+1} \)

b) \( a_n = \frac{1}{n} \sqrt{n^2 + n} \)

c) \( a_n = \frac{1}{n} \sin(n) \)

d) \( a_n = n^3 \cdot 2^{-n} \)

Exercise 9: Convergence of a sequence
Examine the sequence \( (a_n) \) for \( n \geq 1 \) on monotonicity and boundedness. Then draw a conclusion about its convergence:

a) \( a_n = \frac{1-4n}{1+2n} \)

b) \( a_n = \frac{n-1}{2n} \)

c) \( a_n = \frac{2^n + 3^n}{2^n - 3^n} \)

d) \( a_n = \frac{n\sqrt{n} + 10}{n^2} \)

Exercise 10: Series and sigma notation
Fill in the blanks:

<table>
<thead>
<tr>
<th>generating sequence ( a_n )</th>
<th>series ( \sum_{k=0}^{n} a_k )</th>
<th>corresponding function ( f(x) )</th>
<th>Integral ( \int_{1}^{n} f(x) , dx )</th>
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<tr>
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<td>( \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Exercise 11: Arithmetic series
a) Calculate \( \sum_{k=0}^{20} a_k \) and \( \sum_{k=40}^{100} a_k \) for the sequence \( a_n = 2 + \frac{n}{10} \).

b) Calculate \( \sum_{k=0}^{16} a_k \) and \( \sum_{k=40}^{80} a_k \) for the sequence \( (a_n) \) with \( a_0 = 3 \) and \( a_{n+1} = a_n + \frac{1}{2} \).

c) Calculate the initial value \( a_0 \) and the common difference \( d \) for the arithmetic sequence \( (a_n) \) with \( \sum_{k=0}^{10} a_k = 22 \) and \( \sum_{k=5}^{6} a_k = 7 \).

d) Calculate the common difference \( d \) for the arithmetic sequence \( (a_n) \) with \( \sum_{k=10}^{90} a_k = 31 \) and initial value \( a_0 = 1 \).

Exercise 12: Geometric series
a) Calculate \( \sum_{k=0}^{20} a_k \) and \( \sum_{k=70}^{100} a_k \) for the sequence \( a_n = 100 \cdot 0.9^n \).

b) Calculate \( \sum_{k=0}^{10} a_k \) and \( \sum_{k=45}^{50} a_k \) for the sequence \( (a_n) \) with \( a_0 = 3 \) and \( a_{n+1} = 1.2 \cdot a_n \).

c) Calculate initial value \( a_0 \) and common ratio \( q \) for the geometric sequence \( (a_n) \) with \( \sum_{k=0}^{7} a_k = 641 \) and \( \sum_{k=0}^{3} a_k = 625 \).

d) Calculate initial value \( a_0 \) and common ratio \( q \) for the geometric sequence \( (a_n) \) with \( \sum_{k=0}^{4} a_k = 336.16 \) and limit \( \sum_{k=0}^{\infty} a_k = 500 \).
Exercise 13: Limit of a series
Examine the series \( \sum_{k=1}^{n} a_k \) on monotonicity and boundedness. Then draw a conclusion about its convergence:

\[
\begin{align*}
a) \sum_{k=1}^{n} \frac{1}{k^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \\
b) \sum_{k=1}^{n} \frac{1}{(2k+1)^2} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n+1)^2} \\
c) \sum_{k=1}^{n} \frac{1}{3^k} &= \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} \\
d) \sum_{k=1}^{n} \frac{1}{n+k} &= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}
\end{align*}
\]

Exercise 14: Mathematical induction
Prove with mathematical induction
a) \( 2 + 4 + 6 + \ldots + 2n = n(n + 1) \) for \( n \geq 1 \)

b) \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1) \) for \( n \geq 1 \)

c) \( 6 + 24 + 60 + \ldots + n(n + 1)(n + 2) = \frac{1}{4} n(n+1)(n+2)(n+3) \)

d) \( x^0 + x^1 + x^2 + x^3 + \ldots + x^n = \frac{1-x^{n+1}}{1-x} \) for \( n \geq 0 \).

e) \( 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2} \) for \( n \geq 1 \)

f) The sequence \( a_n \) with \( a_1 = 2 \) and \( a_{n+1} = a_n + (n + 1)(n + 2) \) has the general formula \( a_n = \frac{1}{3} n(n+1)(n+2) \).

g) The sequence \( a_n \) with \( a_1 = \frac{3}{4} \) and \( a_{n+1} = a_n - \frac{a_n}{(3n+4)(2n+1)} \) has the general formula \( a_n = \frac{2n+1}{3n+1} \).

h) \( 8^n - 1 \) is divisible by 7 for \( n \geq 1 \)

i) \( n^3 - n \) is divisible by 6 for \( n \geq 2 \)

j) \( (1 + x)^n > 1 + nx \) for \( n \geq 2, x > -1 \) and \( x \neq 0 \) (Bernoulli’s inequality)

Exercise 15: Monotonicity and Boundedness

a) Show by mathematical induction that the sequence \( (a_n) \) with \( a_0 = 3 \) and \( a_{n+1} = \frac{1}{2} \left( a_n + \frac{3}{a_n} \right) \) is positive for all \( n \in \mathbb{N} \)

b) Solve the inequality \( \frac{1}{2} \left( x + \frac{3}{x} \right) > \sqrt{3} \) to \( x \) and show that the sequence from a) has the lower bound \( \sqrt{3} \).

c) Show that the sequence from a) is monotonically decreasing.

d) Find the limit \( a = \lim_{n \to \infty} a_n \) of the sequence from a).

Hint: For \( n \to \infty \) holds \( a_n = a_{n+1} = \lim_{n \to \infty} a_n = a \). Thus in the recursive formula you can plug in \( a_n = a_{n+1} = a \) and the solve to \( a \).
5.7. Solutions to the exercises on sequences and series

Exercise 1: Linear and exponential growth
All lengths are given in dm, all areas in dm²:

a) \( U_0 = 4, U_1 = 6, U_2 = 8, U_3 = 10 \) and \( U_4 = 12 \)

b) \( U_{n+1} - U_n = 2 \) (linear growth)

c) \( U_{n+1} = U_n + 2 \) (recursive formula)

d) \( U_n = 4 + 2n \) (general formula)

e) \( A_0 = 1, A_1 = 1 + \frac{1}{3} \approx 1.33, A_2 = 1 + \frac{1}{3} + \frac{1}{9} \approx 1.44, A_3 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \approx 1.48 \)

and \( A_4 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \approx 1.49 \)

f) \( A_{n+1} - A_n = 3^{n+1}, \left( \frac{1}{3} \right)^{n+1} = \left( \frac{1}{3} \right)^{n+1} \) (exponential growth)

g) \( A_{n+1} = A_n + \left( \frac{1}{3} \right)^{n+1} \) (recursive formula)

h) \( A_n = 1 + \left( \frac{1}{3} \right)^1 + \left( \frac{1}{3} \right)^2 + \ldots + \left( \frac{1}{3} \right)^n = \frac{1 - 1/3^{n+1}}{1 - 1/3} = \frac{3}{2} \left( 1 - \left( \frac{1}{3} \right)^{n+1} \right) = \frac{3}{2} - \frac{3}{2} \left( \frac{1}{3} \right)^n \) (general formula)

i) \( U_{100} = 204 \) and \( A_{100} \approx 1,5 \) ⇒ The circumference grows beyond all bounds but the area is bounded!

j) \( \lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{3}{2} \left( 1 - \left( \frac{1}{3} \right)^n \right) = \frac{3}{2} \)

Exercise 2: Calculating terms with recursive and general formulae

a) \( a_0 = 100; a_1 = 50; a_2 = 25; a_3 = 12.5; a_4 = 6.25 \)

b) \( a_0 = 50; a_1 = 75; a_2 = 87.5; a_3 = 93.75; a_4 = 96.875 \)

c) \( a_0 = 1; a_1 = \frac{1}{2}; a_2 = \frac{1}{3}; a_3 = \frac{1}{4}; a_4 = \frac{1}{5} \)

d) \( a_0 = 2; a_1 = 6; a_2 = 12; a_3 = 30 \) and no limit

e) \( a_0 = 3; a_1 = 3.5; a_2 = 4; a_3 = 4.5; a_4 = 5 \)

f) \( a_0 = 3; a_1 = 4.5; a_2 = 6.75; a_3 = 10.125; a_4 = 15.1875 \)

g) \( a_0 = 3; a_1 = 4; a_2 = 4.5; a_3 = 4.75; a_4 = 4.875 \)

h) \( a_0 = 3; a_1 = 3.3; a_2 = 3.74; a_3 = 3.98; a_4 = 4.18 \)

Exercise 3: Finding general and recursive formulae from given terms of a sequence

a) \( a_{n+1} = a_n + 2 \) with \( a_0 = 1 \) ⇒ \( a_n = 1 + 2n \)

b) \( a_{n+1} = 2a_n \) with \( a_0 = 3 \) ⇒ \( a_n = 3 \cdot 2^n \)

c) \( a_{n+1} = 3a_n \) with \( a_0 = 2 \) ⇒ \( a_n = 2 \cdot 3^n \)

d) \( a_{n+1} = a_n + 2n + 3 \) with \( a_0 = 2 \) ⇒ \( a_n = n^2 + 1 \)

Exercise 4: Converting general formulae into general formulae

a) \( a_{n+1} = a_n + 3 \) with \( a_0 = 2 \)

b) \( a_{n+1} = a_n + 2n - 1 \) with \( a_0 = 0 \)

c) \( a_{n+1} = a_n + 1 \) with \( a_0 = 1 \)

d) \( a_{n+1} = a_n + \frac{1}{n+1(n+2)} \) with \( a_0 = 0 \)

Exercise 5: Converting recursive formulae into general formulae

a) \( a_n = 3^n + 2 \)

b) \( a_n = 20 \cdot 0.8^n \)

c) \( a_n = n(n+1) \)

d) \( a_n = n^2 \)
Exercise 6: Monotonicity of a sequence

a) monotonically increasing, since \( \frac{a_{n+1}}{a_n} = \frac{n(n+1)}{(n+2)(n-1)} = \frac{n^2 + n}{n^2 + n - 2} > 1 \) for all \( n \in \mathbb{N} \).

b) monotonically increasing, since \( \frac{a_{n+1}}{a_n} = \frac{\sqrt{(n+1)^2 - (n+1)}}{\sqrt{n^2 - n}} = \frac{n^2 + n}{\sqrt{n^2 - n}} > 1 \) for all \( n \in \mathbb{N} \).

(c) monotonically increasing for \( n \geq 2 \),

since \( a_{n+1} - a_n = (n+1)^3 - 3(n+1)^2 - n^3 + 3n^2 = 3n^2 - 3n - 2 = 3(n^2 - n - \frac{2}{3}) > 0 \) for \( n \geq 2 \)

d) monotonically decreasing for \( n \geq 2 \),

since \( a_{n+1} - a_n = (n+1)^2 - 2^{-(n+1)} - n^2 \cdot 2^n = 2^{-(n+1)}(n^2 + 2n + 1 - 2n^2) = 2^{-(n+1)}(n^2 + 2n + 1) < 0 \) for \( n \geq 2 \)

Exercise 7: Boundedness of a sequence

a) Upper bound \( S_n = 1 \), since \( a_n \leq 1 \) \( \Rightarrow \frac{n}{n+1} \leq 1 \) for all \( n \in \mathbb{N} \) and lower bound \( S_n = 0 \), since \( a_n \geq 0 \) \( \Rightarrow \frac{n}{n+1} \geq 0 \) for all \( n \in \mathbb{N} \).

b) No upper bound \( S \), since there is no \( S \in \mathbb{R} \) with \( a_n \leq S \) \( \Rightarrow \sqrt{n^2 + n} \leq S \) \( \Rightarrow n^2 + n \leq S^2 \) for all \( n \in \mathbb{N} \).

Lower bound \( S_n = 0 \), since \( a_n \geq 0 \) \( \Rightarrow \sqrt{n^2 + n} \geq 0 \) for all \( n \in \mathbb{N} \).

c) Upper bound \( S_n = 0 \), since \( a_n \leq 0 \) \( \Rightarrow n^2 - n^3 \leq 0 \) \( \Rightarrow n^2(1 - n) \leq 0 \) for all \( n \in \mathbb{N} \).

No lower bound \( S \), since there is no \( S \in \mathbb{R} \) with \( a_n \geq S \) \( \Rightarrow n^2 - n^3 \geq S \) for all \( n \in \mathbb{N} \).

d) Upper bound \( S_n = 1 \), since \( a_n \leq 1 \) \( \Rightarrow n^2 \cdot 3^{-n} \leq 1 \) \( \Rightarrow n^2 \leq 3^a \) for all \( n \in \mathbb{N} \).

Lower bound \( S_n = 0 \), since \( n^2 \cdot 3^{-n} \geq 0 \) for all \( n \in \mathbb{N} \).

Exercise 8: Limit of a sequence

We have to show that for any \( \varepsilon > 0 \) there is a \( n_0 \) so that the condition \( |a_n - a| < \varepsilon \) holds for all \( n > n_0 \).

a) \( \lim_{n \to \infty} a_n = 1 \), since \( |a - a_n| \leq \varepsilon \) \( \Leftrightarrow |1 - \frac{n+2}{n+1}| \leq \varepsilon \) \( \Leftrightarrow |\frac{1}{n+1}| \leq \varepsilon \) \( \Leftrightarrow \frac{1}{n+1} \leq \varepsilon \) holds for all \( n \geq n_0 = \frac{1}{\varepsilon} - 1 \).

b) \( \lim_{n \to \infty} a_n = 1 \), since \( |a - a_n| \leq \varepsilon \) \( \Leftrightarrow |1 - \frac{1}{\sqrt{n^2 + n}}| \leq \varepsilon \) \( \Leftrightarrow 1 - \sqrt{1 + \frac{1}{n}} \leq \varepsilon \) \( \Leftrightarrow \sqrt{1 + \frac{1}{n}} \leq 1 - \varepsilon \)

\( \Rightarrow 1 + \frac{1}{n} \leq (1 - \varepsilon)^2 \Rightarrow \frac{1}{1 - (1 - \varepsilon)} \leq n \Rightarrow \frac{\varepsilon}{(2 - \varepsilon)} \leq n \) holds for all \( n \geq n_0 = \frac{\varepsilon}{(2 - \varepsilon)} \).

c) \( \lim_{n \to \infty} a_n = 0 \), since \( |a - a_n| \leq \varepsilon \) \( \Leftrightarrow |\frac{1}{n} \sin(n)| \leq \varepsilon \) \( \Leftrightarrow \frac{1}{n} \leq \varepsilon \) holds for all \( n \geq n_0 = \frac{1}{\varepsilon} \).

d) \( \lim_{n \to \infty} a_n = 0 \), since \( |a - a_n| \leq \varepsilon \) \( \Leftrightarrow |n^2 \cdot 2^{-n}| \leq \varepsilon \) \( \Leftrightarrow \frac{1}{n^2} \leq \frac{2^n}{\varepsilon} \) holds for all \( n \geq 10 \).

Exercise 9: Convergence of a sequence

a) Boundedness: \( a_n = \frac{1 - 4n}{1 + 2n} = -2 + \frac{3}{1 + 2n} \Rightarrow -2 < a_n < 1 \), since \( a_n + 2 = \frac{3}{1 + 2n} \) and \( 0 < \frac{3}{1 + 2n} < 3 \)

Monotonicity: \( a_{n+1} - a_n = \frac{3}{1 + 2(n+1)} - \frac{3}{1 + 2n} = \frac{3}{2n + 3} - \frac{3}{2n + 1} = 0 \Rightarrow (a_n) \) decreases monotonically (to \( \lim_{n \to \infty} a_n = -2 \))

b) Boundedness: \( a_n = \frac{n - 1}{2n} = \frac{1}{2} - \frac{1}{2n} \Rightarrow 0 < a_n < \frac{1}{2} \), since \( 0 < \frac{1}{2n} < \frac{1}{2} \).

Monotonicity: \( a_{n+1} - a_n = -\frac{1}{2(n+1)} + \frac{1}{2n} > 0 \Rightarrow (a_n) \) increases monotonically (to \( \lim_{n \to \infty} a_n = \frac{1}{2} \))
c) Boundedness: \(a_n = \frac{2^n + 3^n}{2^n - 3^n} = \frac{3^2 \left( \frac{n}{3} + 1 \right)}{3^2 \frac{n}{3} - 1} \Rightarrow -2 < a_n < 0, \text{ since } 0 < \frac{1 + \frac{2^n}{3^2}}{1 - \frac{2^n}{3^2}} < 1 + \frac{2^n}{3^2} < 2\)

Monotonicity: \(a_{n+1} - a_n = \frac{2^{n+1} + 3^{n+1}}{2^{n+1} - 3^{n+1}} - \frac{2^n + 3^n}{2^n - 3^n} = \frac{2^n(2^{n+1} - 3^{n+1}) + 3^{n+1}(2^n - 3^n)}{(2^{n+1} - 3^{n+1})(2^n - 3^n)} = \frac{2^{n+1} - 2^n + 3^{n+1} - 3^n}{(2^{n+1} - 3^{n+1})(2^n - 3^n)} \Rightarrow (a_n) \text{ increases monotonically (to } \lim_{n \to \infty} a_n = -1)\)

d) Boundedness: \(a_n = \frac{\sqrt{n^2} + 10}{n^2} = \frac{1}{\sqrt{n} + 10} \Rightarrow 0 < a_n < 1 + 10 = 11\)

Monotonicity: \(a_{n+1} - a_n = \frac{1}{\sqrt{n + 1}} + \frac{10}{(n + 1)^2} - \frac{1}{\sqrt{n}} - \frac{10}{n^2} = \left( \frac{1}{\sqrt{n + 1}} - \frac{1}{\sqrt{n}} \right) + \frac{10}{(n + 1)^2} - \frac{10}{n^2} < 0, \text{ since both brackets < 0}\)

\(\Rightarrow (a_n) \text{ decreases monotonically (to } \lim_{n \to \infty} a_n = 0)\)

**Exercise 10: Series and sigma notation**

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<td>(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots)</td>
<td>(\frac{1}{x})</td>
<td>(\ln(n + 1))</td>
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<tr>
<td>(\frac{1}{3^n})</td>
<td>(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots)</td>
<td>(\frac{1}{3^n})</td>
<td>(3 \cdot \ln(3) \left( 1 - \frac{1}{3^n} \right))</td>
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<tr>
<td>(\frac{1}{n^2})</td>
<td>(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots)</td>
<td>(\frac{1}{x^2})</td>
<td>(1 - \frac{1}{n + 1})</td>
</tr>
<tr>
<td>(\frac{1}{n(n+1)})</td>
<td>(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots)</td>
<td>(\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1})</td>
<td>(\ln(2) + \ln \left( \frac{n + 1}{n + 2} \right))</td>
</tr>
</tbody>
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**Exercise 11: Arithmetic series**

a) \(\sum_{k=0}^{100} a_k = 63\) and \(\sum_{k=1}^{100} a_k = \sum_{k=0}^{99} a_k = 707 - 297 = 410\).

b) \(\sum_{k=0}^{16} a_k = 119\) and \(\sum_{k=0}^{80} a_k = \sum_{k=0}^{79} a_k = 1863 - 510 = 1353\).

c) \(\sum_{k=0}^{9} a_k = 110d + 55d = 22 \Rightarrow a_0 + 5d = 2\) and \(\sum_{k=0}^{9} a_k = 7a_0 + 21d = 7 \Rightarrow a_0 + 3d = 1\) result in \(a_0 = -\frac{1}{2}\) and \(d = \frac{1}{2}\)

d) \(\sum_{k=0}^{90} a_k = 91 - 1 + 4095d = \sum_{k=0}^{9} a_k + 31 \cdot 10 - 1 + 45d + 31 \approx 91 + 4095d = 41 + 45d \approx 50 = 4050\) d result in \(d = \frac{1}{81}\)

**Exercise 12: Geometric series**

a) \(\sum_{k=0}^{20} a_k = 1000(1 - 0.9^{20}) = 890,581\) and \(\sum_{k=0}^{100} a_k = 1000(1 - 0.9^{100}) = 999,976\).

b) \(\sum_{k=0}^{12} a_k = 15 - (1.2^{12} - 1) \approx 96,45\) and \(\sum_{k=0}^{90} a_k = 15 - (1.2^{91} - 1) \approx 163,792,89\)

c) \(\sum_{k=0}^{6} a_k = a_0 \frac{q^7 - 1}{q - 1} = 641\) and \(\sum_{k=0}^{\infty} a_k = a_0 \frac{q^4 - 1}{q - 1} = 625\) result in \(\frac{q^8 - 1}{q^4 - 1} = \frac{641}{625} \approx (q^4 - 1)(q^4 + 1) \approx 1.0256\)

\(\Rightarrow q^4 = 0.256 \Rightarrow common\ ratio\ q = 0.4\) and initial value \(a_0 \approx 384.85\)

d) \(\sum_{k=0}^{4} a_k = a_0 \frac{q^5 - 1}{q - 1} = 336.16\) and limit \(\sum_{k=0}^{\infty} a_k = a_0 \frac{1}{1 - q} = 500\) result in \(q^5 - 1 = 336.16 \approx 500 \Rightarrow q^5 = 0.32768 \Rightarrow common\ ratio\ q = 0.8\) and initial value \(a_0 = 100\).
Exercise 13: Limit of a series

a) \( s_n = \sum_{k=1}^{n} \frac{1}{k^2} \) is monotonically increasing, since \( s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0 \) for all \( n \in \mathbb{N} \). \((s_n)\) is bounded above, since
\[
 s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 1 + \int_1^n \frac{1}{x^2} \, dx = 1 + \left[-\frac{1}{x}\right]_1^n = 2 - \frac{1}{n} < 2 \text{ for all } n \in \mathbb{N}. \text{ Therefore } (s_n) \text{ converges to a limit } \lim_{n \to \infty} s_n \leq 2. \text{ L. Euler showed in 1736, that } \lim_{n \to \infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.64
\]

b) \( s_n = \sum_{k=0}^{n} \frac{1}{(2k+1)^2} \) is monotonically increasing, since \( s_{n+1} - s_n = \frac{1}{(2n+3)^2} > 0 \) for all \( n \in \mathbb{N} \). \((s_n)\) is bounded above, since
\[
 s_n = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n)^2} < 1 + \int_1^n \frac{1}{x^2} \, dx = 1 + \left[-\frac{1}{x}\right]_1^n = 2 - \frac{1}{2n+1} < \frac{3}{2} \text{ for all } n \in \mathbb{N}. \text{ Therefore } (s_n) \text{ converges to } \lim_{n \to \infty} s_n \leq \frac{3}{2}. \text{ L. Euler showed, that } \lim_{n \to \infty} s_n = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \approx 1.23
\]

c) \( s_n = \sum_{k=0}^{n} \frac{1}{3^k} \) is monotonically increasing, since \( s_{n+1} - s_n = \frac{1}{3^n} > 0 \) for all \( n \in \mathbb{N} \). \((s_n)\) is bounded above, since
\[
 s_n = 1 + \frac{1}{3} + \frac{1}{9} + \ldots + \frac{1}{3^n} < \int_{-1}^n e^{-\ln(3)x} \, dx = \left[-\frac{1}{\ln(3)} e^{-\ln(3)x}\right]_{-1}^n = \frac{3}{\ln(3)} - \frac{1}{\ln(3)} \cdot 3^n < \frac{3}{\ln(3)} \text{ for } n \in \mathbb{N}. \text{ Therefore } (s_n) \text{ converges to } \lim_{n \to \infty} s_n \leq \frac{3}{\ln(3)}. \text{ The exact value of this limit can be obtained by using the summation rule for geometric series: } \sum_{k=0}^{\infty} \frac{1}{3^k} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{3^k} = \lim_{n \to \infty} 1 - \frac{(1/3)^{n+1}}{1 - 1/3} = \frac{1}{2}.
n
\]

d) \( s_n = \sum_{k=1}^{n} \frac{1}{n+k} \) is monotonically increasing, since \( s_{n+1} - s_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \) for \( n \in \mathbb{N} \). \((s_n)\) is bounded above, since
\[
 s_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+k} < \int_0^n \frac{1}{n+x} \, dx = \ln(n+x) \big|_0^n = \ln(2n) - \ln(n) = \ln(2)
\]
for all \( n \geq 1 \) and \( s_0 = 1 \). Therefore \((s_n)\) converges to \( \lim_{n \to \infty} s_n \leq \ln(2) \).

Exercise 14: Mathematical induction

a) Base case \( n = 1: 2 = 1(1+1) \)

Inductive step \( n \Rightarrow n + 1: \)

Inductive hypothesis for some \( n: 2 + 4 + 6 + \ldots + 2n = n(n + 1) \)

We have to show the statement for \( n + 1: 2 + 4 + 6 + \ldots + 2n + 2(n + 2) = (n + 1)(n + 2) \)

Plug the hypothesis into the left side:
\[
2 + 4 + 6 + \ldots + 2n + 2(n + 2) = n(n + 1) + 2n + 2 = n^2 + 3n + 2
\]

Right side: \( (n + 1)(n + 2) = n^2 + 3n + 2 \) qed

b) Base case \( n = 1: 1^2 = \frac{1}{6} \cdot 1(1+1)(2+1) = 1 \)

Inductive step \( n \Rightarrow n + 1: \)

Inductive hypothesis for some \( n: 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6} n(n + 1)(2n + 1) \)

We have to show the statement for \( n + 1: 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n + 1)^2 = \frac{1}{6} (n + 1)(n + 2)(2n + 3) \)

Plug the hypothesis into the left side:
\[
1^2 + 2^2 + 3^2 + \ldots + n^2 + (n + 1)^2 = \frac{1}{6} n(n + 1)(2n + 1) + (n + 1)^2 = \frac{1}{3} n^3 + \frac{3}{2} n^2 + \frac{13}{6} n + 1
\]

Right side: \( \frac{1}{6} (n + 1)(n + 2)(2n + 3) = \frac{1}{6} (2n^3 + 9n^2 + 13n + 6) = \frac{1}{3} n^3 + \frac{3}{2} n^2 + \frac{13}{6} n + 1 = \text{left side. qed} \)
c) Base case $n = 1$: $6 = \frac{1}{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 6$

Inductive step $n \Rightarrow n + 1$:

Inductive hypothesis for some $n$: $6 + 24 + 60 + \ldots + n(n + 1)(n + 2) = \frac{1}{4} n(n + 1)(n + 2)(n + 3)$

We have to show the statement for $n + 1$:

$6 + 24 + 60 + \ldots + n(n + 1)(n + 2) + (n + 1)(n + 2)(n + 3) = \frac{1}{4} (n + 1)(n + 2)(n + 3)(n + 4)$

Plug the hypothesis into the left side:

$6 + 24 + 60 + \ldots + n(n + 1)(n + 2) + (n + 1)(n + 2)(n + 3) = \frac{1}{4} n(n + 1)(n + 2)(n + 3) + (n + 1)(n + 2)(n + 3)$

$= \frac{1}{4} (n + 1)(n + 2)(n + 3)(n + 4) = \text{right side, qed.}$

d) Base case $n = 0$: $1 = \frac{1}{1-x}$

Inductive step $n \Rightarrow n + 1$:

Inductive hypothesis for some $n$: $x^0 + x^1 + x^2 + x^3 + \ldots + x^n = \frac{1- x^{n+1}}{1-x}$

We have to show the statement for $n + 1$: $x^0 + x^1 + x^2 + x^3 + \ldots + x^n + x^{n+1} = \frac{1- x^{n+2}}{1-x}$

Plug the hypothesis into the left side:

$x^0 + x^1 + x^2 + x^3 + \ldots + x^n + x^{n+1} = \frac{1}{1-x} + x^{n+1} = \frac{1- x^{n+1} + x^{n+1} - x^{n+2}}{1-x} = \frac{1- x^{n+2}}{1-x} = \text{right side, qed}$

e) Base case $n = 1$: $1 = \frac{1-2x + x^2}{(1-x)^2} = 1$

Inductive step $n \Rightarrow n + 1$:

Inductive hypothesis for some $n$: $1 + 2x + 3x^2 + 4x^3 + \ldots + nx^n = \frac{1-(n+1)x^{n+1} + nx^{n+1}}{(1-x)^2}$

We have to show the statement for $n + 1$: $1 + 2x + 3x^2 + 4x^3 + \ldots + nx^n + (n+1)x^n = \frac{1-(n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}$

Plug the hypothesis into the left side:

$\frac{1}{(1-x)^2} - \frac{(n+1)x^n + nx^{n+1}}{(1-x)^2} + (n+1)x^n = \frac{1-(n+1)x^n + nx^{n+1} + (1-x)^2(n+1)x^n}{(1-x)^2}$

$= \frac{1-(n+1)x^n + nx^{n+1} + (n+1)x^n + 2(n+1)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}$

$= \frac{1-(n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2} = \text{right side, qed}$

f) Base case $n = 1$: $2 = \frac{1}{3} \cdot 1 \cdot 2 \cdot 3 = 2$

Inductive step $n \Rightarrow n + 1$:

Inductive hypothesis for some $n$: $a_n = \frac{1}{3} n(n + 1)(n + 2)$

We have to show the statement for $n + 1$: $a_{n+1} = \frac{1}{3} (n + 1)(n + 2)(n + 3)$

Plug the hypothesis into the left side:

$a_{n+1} = a_n + (n+ 1)(n + 2) = \frac{1}{3} n(n + 1)(n + 2) + (n + 1)(n + 2) = \frac{1}{3} n^3 + n^2 + \frac{2}{3} n + n^2 + 3n + 2 = \frac{1}{3} n^3 + 2n^2 + \frac{11}{3} n + 2$

Right side: $\frac{1}{3} (n + 1)(n + 2)(n + 3) = \frac{1}{3} (n^3 + 3n^2 + 2n^2 + 3n + 2) = \frac{1}{3} n^3 + 2n^2 + \frac{11}{3} n + 2 = \text{left side, qed.}$
g) **Base case for** \( n = 1: \frac{3}{4} = \frac{2+1}{3+1} = \frac{3}{4} \)

**Inductive step** \( n \to n + 1:\)

**Inductive hypothesis for some** \( n: a_n = \frac{2n+1}{3n+1} \)

*We have to show the statement for* \( n + 1: a_{n+1} = \frac{2n+3}{3n+4} \)

*Plug the hypothesis into the left side:*

\[
a_{n+1} = a_n - \frac{a_n}{(3n+4)(2n+1)} = \frac{2n+1}{3n+1} \left(1 - \frac{1}{(3n+4)(2n+1)}\right) = \frac{2n+1}{3n+1} \left(1 + \frac{6n^2 + 13n + 3}{(3n+4)(2n+1)}\right) = \frac{2n+1}{3n+1} \left(\frac{(3+1)(2n+3)}{3n+4}(2n+1)\right) = \frac{2n+3}{3n+4}
\]

\( = \) right side. \( \text{qed.} \)

**h) Base case** \( n = 1: 8^1 - 1 = 7 \) obviously is divisible by 7

**Inductive step** \( n \Rightarrow n + 1:\)

**Inductive hypothesis for some** \( n: 8^n - 1 \) is divisible by 7

*We have to show the statement for* \( n + 1: 8^{n+1} - 1 = 8 \cdot 8^n - 1 = 7 \cdot 8^n + 8^n - 1 \) is divisible by 7

Obviously the first summand \( 7 \cdot 8^n \) is divisible by 7. **According to the inductive hypothesis** the second summand \( 8^n - 1 \) too is divisible by 7 and therefore the complete sum is divisible by 7. \( \text{qed.} \)

**i) Base case** \( n = 1: 2^3 - 2 = 6 \) is divisible by 6

**Inductive step** \( n \Rightarrow n + 1:\)

**Inductive hypothesis for some** \( n: n^3 - n \) is divisible by 6

*We have to show the statement for* \( n + 1:\)

\[
(n + 1)^3 - (n + 1) = n^3 + 3n^2 + 2n = (n^3 - n) + (3n^2 + 3n) = (n^3 - n) + 6 \cdot \frac{1}{2} n(n + 1) \text{ is divisible by 6}
\]

**According to the inductive hypothesis** the left summand \( n^3 - n \) is divisible by 6. But since either \( n \) or \( n + 1 \) is even, \( \frac{1}{2} n(n + 1) \) is an integer. Therefore the right summand \( 6 \cdot \frac{1}{2} n(n + 1) \) too is divisible by 6 and so is the complete sum. \( \text{qed.} \)

**j) Base case** \( n = 2: (1 + x)^2 = 1 + 2x + x^2 > 1 + 2x, \text{ since } x^2 > 0 \)

**Inductive step** \( n \Rightarrow n + 1:\)

**Inductive hypothesis for some** \( n: (1 + x)^n > 1 + nx \)

*We have to show the statement for* \( n + 1: (1 + x)^{n+1} > 1 + (n + 1)x \)

*Plug the hypothesis into the left side:

\[
(1 + x)^{n+1}(1 + x) > (1 + nx)(1 + x), \text{ since we assume } 1 + x > 0
\]

\[
= 1 + (n + 1)x + x^2
\]

\[
> 1 + (n + 1)x, \text{ since } x^2 > 0
\]

\[
= \text{right side. \( \text{qed.} \)}
\]

**Aufgabe 15: Mathematical induction, Monotonicity and Boundedness**

**a) Base case** \( n = 0: a_0 = 3 > 0 \)

**Inductive step** \( n \Rightarrow n + 1:\)

**Inductive hypothesis for some** \( n: a_n > 0 \)

*We have to show the statement for* \( n + 1: a_{n+1} > 0 \)

*Use the hypothesis: \( a_{n+1} = \frac{1}{2} \left( a_n + \frac{3}{a_n} \right) > 0, \text{ since } a_n > 0, \text{ qed.} \)

**b) \( \frac{1}{2} (x + \frac{3}{x}) > \sqrt{3} \iff x + \frac{3}{x} > 2 \sqrt{3} \iff x^2 + 3 > 2 \sqrt{3} x \iff x^2 - 2 \sqrt{3} x + 3 > 0 \iff (x - \sqrt{3}) > 0 \text{ for } x \in \mathbb{R}. \)

**c) \( a_{n+1} - a_n = \frac{1}{2} \left( a_n + \frac{3}{a_n} \right) - a_n = \frac{1}{2} \left( \frac{3 - a_n^2}{a_n} \right) > 0, \text{ because in b) we have shown } a_n^2 < 3 \Rightarrow (a_n) \text{ is monotonically decreasing.} \)

**d) For** \( n \to \infty \text{ we have } a_n = a_{n+1} \lim_{n \to \infty} a_n = a \text{ and so } a = \frac{1}{2} \left( a + \frac{3}{a} \right) \iff \frac{1}{2} a = 1 \pm \frac{3}{a} \iff a^2 = 3 \iff a = \text{ if } a > 0 \text{ because of } a) \)

\( = \lim_{n \to \infty} a_n = \sqrt{3}. \)