5.7. Sequences and series

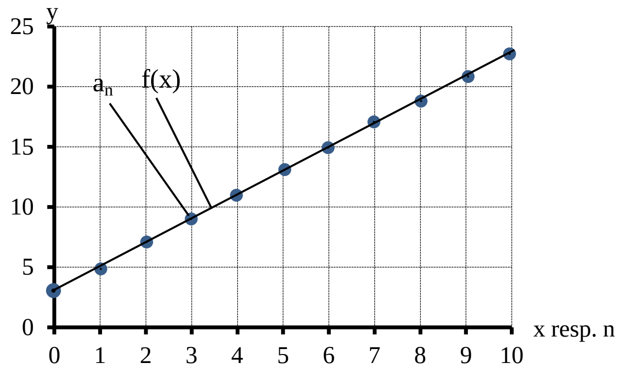
Natural processes like growth of populations of particles, cells or animals, but also share values, prices, exchange rates, sales numbers etc. evolve in many little **steps**. Every single chemical reaction, cell division, birth, death, sale and buy action produces a small **jump** of the population or rate. In a mathematical context, these steps are **discontinuities** incommensurable with the **continuous functions** (see 4.7. and 5.6.) usually used to model these processes. Instead of continuous functions **discrete sequences and series** provide more realistic simulations of these processes.

5.7.1. Recursive and general formulae for sequences

Example: Exercises on sequences and series No. 1

Definition

A function with domain D = ℕ = {0; 1; 2; ...} is called a **sequence**. Instead of the variable x we use the **index n**; the value f(x) of f at x becomes the **n^{th} term** a_n.



Example:

f(x) = 2x + 3 with domain D = ℝ is a function

a_n = 2n + 3 with domain D = ℕ is a sequence

Exercises on sequences and series No. 2

Definition

1. A **general** formula gives the value of a_n **depending only on n**. As in a function f(x) the general formula makes it possible to find the value of a_n simply by plugging in n without having to know the value of the previous terms.
2. A **recursive** formula gives the value of a_{n+1} **depending on the previous term a_n**. Because a_n depends on a_{n-1} and so on actually **all previous term have to be known**. On the other hand the recursive formula often is much simpler and easier to understand than the general formula which may even be **unknown**!

Example: Recursive and general formula for exponential growth (see 4.7.4.)
An amount of a_0 = 1000 € is deposited in a savings account with a compound interest of 15 %. Give a recursive formula and a general formula for the amount after n years.

Solution:
recursive: $a_{n+1} = \cdot a_n + 0{,}15\, a_n = 1{,}15\cdot a_n$ with $a_0 = 1000$
general: $a_n = 1{,}15\cdot a_{n-1} = 1{,}15^2\cdot a_{n-2} = \ldots = 1{,}15^n\cdot a_0 = 1{,}15^n\cdot 1000$.

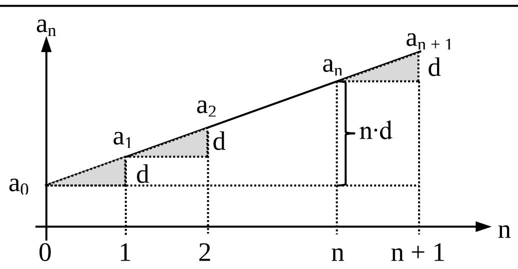
Examples for sequences used as approximation for irrational numbers in analysis:

1. **recursive** approximation of a squareroot $\sqrt{a} = \lim\limits_{n\to\infty} x_n$ with $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ (see 5.7.5)
2. **recursive** approximation of π = $\lim\limits_{n\to\infty} x_n$ with $\pi_{2n} = n\sqrt{2-2\sqrt{1-\left(\frac{\pi_n}{n}\right)^2}}$ (see 5.7.6 and 5.7.7)
3. **general** approximation of Euler’s number e = $\lim\limits_{n\to\infty} e_n$ with $e_n = \left(1+\frac{1}{n}\right)^n$ (see 5.7.8)
4. **recursive** approximation of a zero (Newton-Raphson-method) x = $\lim\limits_{n\to\infty} x_n$ with $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (see 5.3.6)

Arithmetic sequences
A sequence with
- the **recursive formula** $a_{n+1} = d + a_n$
- the **general formula** $a_n = n\cdot d + a_0$

describes a linear growth with
- **initial value** a_0
- **common difference** d

and is called a **arithmetic sequence**.



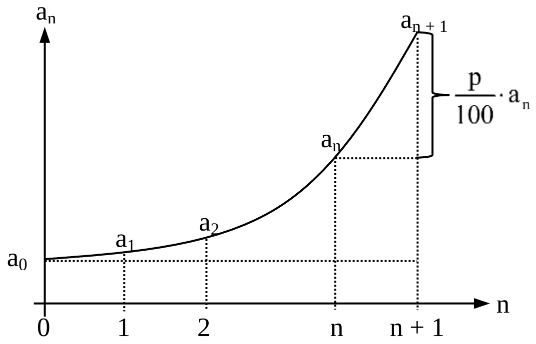
Geometric sequences
A sequence with

- the **recursive formula** $a_{n+1} = \left(1+\frac{p}{100}\right)\cdot a_n$
- the **general formula** $a_n = \left(1+\frac{p}{100}\right)^n \cdot a_0$

describes an exponential growth with

- **initial value** a_0
- **percentage p**
- **common ratio** $q = \left(1+\frac{p}{100}\right)$

and is called a **geometric sequence**.



Exercises on sequences and series No. 3

Example for converting a general formula into a recursive formula
Fin the recursive formula of $a_n = n^3 - 3n^2$.

Solution:
$a_{n+1} - a_n = [(n + 1)^3 - 3(n + 1)^2] - [n^3 - 3n^2]$
$= n^3 + 3n^2 + 3n + 1 - 3n^2 - 6n - 3 - n^3 + 3n^2$
$= 3n^2 - 3n - 2$

$\Rightarrow a_{n+1} = a_n + 3n^2 - 3n - 2$ mit $a_0 = 0$

Exercises on sequences and series No. 4

A formula for the first n natural numbers = summation rule of the arithmetic series (see 5.7.4.) with $a_n = n$:

$1 + 2 + 3 + ... + (n - 2) + (n - 1) + n = (n + 1) + (n + 1) + (n + 1) + ... = \frac{1}{2} n\cdot(n + 1)$, because

by pairing numbers with equal sums you get $\frac{1}{2}n$ pairs with sum $(n + 1)$, if n is even

or $\frac{1}{2}(n - 1)$ pairs with sum $(n + 1)$ and the lone value $\frac{1}{2}(n + 1)$ in the middle, if n is odd

Example for the conversion of a recursive formula into a general formula
Find a general formula for the sequence $a_{n+1} = a_n + 4n - 2$ with $a_0 = 1$.

Solution:
$a_n = a_{n-1} + 4(n - 1) - 2$
$= a_{n-1} + 4n - 6$
$= a_{n-2} + 4(n - 1) + 4n - 6 - 6$
$= a_{n-3} + 4(n - 2) + 4(n - 1) + 4n - 6 - 6 - 6$
.
.
.
$= a_0 + 4(1 + 2 + 3 + ... + n) - n\cdot 6 = 4\cdot \frac{1}{2} n\cdot(n + 1) - 6n$ | **apply summation rule from above**
$= 1 + 2n^2 + 2n - 6n$
$= 1 + 2n^2 - 4n$
$= 2n(n - 2) + 1$

Exercises on sequences and series No 5

5.7.2. Monotone and bounded sequences (see 5.3.1.)

Definition
A sequence (a_n) is called

- **monotonically increasing**, if $a_{n+1} \geq a_n \Leftrightarrow a_{n+1} - a_n \geq 0$ (**difference criterion**)

  for **positive** terms a_n and a_{n+1} the condition $\frac{a_{n+1}}{a_n} \geq 1$ is equivalent (**ratio criterion**)

- **monotonically decreasing**, if $a_{n+1} \leq a_n \Leftrightarrow a_{n+1} - a_n \leq 0 \Leftrightarrow \frac{a_{n+1}}{a_n} \leq 1$ viz for positive a_n and a_{n+1} we have $\frac{a_{n+1}}{a_n} \leq 1$.

for all $n \in \mathbb{N}$. The **ratio criterion** is convenient in case of a_n being a product.

Example 1:
Examine the sequence $a_n = n^3 - 16n$ on monotonic behaviour

Solution
Since a_n is a **sum** the **difference criterion** is suitable $a_{n+1} - a_n \geq 0$

$$\begin{aligned} a_{n+1} - a_n &= n^3 + 3n^2 + 3n + 1 - 16n - 16 - n^3 + 16n \\ &= 3n^2 + 3n - 15 \\ &= 3(n^2 + n - 5) \\ &> 0 \text{ für } n \geq 2 \end{aligned}$$

$\Rightarrow a_n$ (strictly) monotonically increases for $n \geq 2$.

Example 2:
Examine the sequence $a_n = n^4 \cdot 2^{-n}$ on monotonic behaviour

Solution

Since a_n is a **product and always positive** the **ratio criterion** is convenient: $\frac{a_{n+1}}{a_n} \geq 1$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^4 \cdot 2^{-(n+1)}}{n^4 \cdot 2^{-n}} \\ &= \frac{(n+1)^4}{n^4 \cdot 2} \\ &= \frac{1}{2} \cdot \left(\frac{n+1}{n}\right)^4 \\ &= \underbrace{\frac{1}{2} \cdot \underbrace{\left(1 + \frac{1}{n}\right)^4}_{\to 1 \text{ for } n \to \infty}}_{\to \frac{1}{2} \cdot 1 = \frac{1}{2} \text{ for } n \to \infty}. \end{aligned}$$

With the help of the GDC in **sequential mode** we quickly calculate some values: $\frac{a_6}{a_5} \approx 1{,}04 > 1$ und $\frac{a_7}{a_6} \approx 0{,}93 < 1$

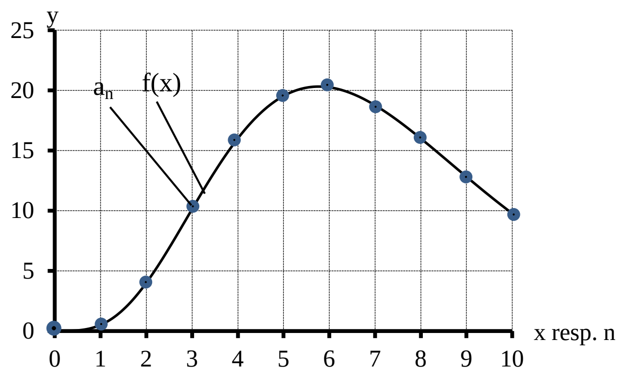
$\Rightarrow a_n$ is (strictly) monotonically decreasing for $n \geq 6$.

Exercises on sequences and series No. 6

Definition
A sequence (a_n) is
- **bounded above**, if $a_n \leq S$
- **bounded below**, if $a_n \geq S$

for an **upper resp. lower bound** $S \in \mathbb{R}$ and all $n \in \mathbb{N}$

Example:
Examine if the sequence $a_n = n^4 \cdot 2^{-n}$ is bounded and give reasons.



Solution:

a_n is **bounded below**,
since $a_n > 0$ for all $n \in \mathbb{N}$.

a_n is **bounded above**,
since it **decreases** after the maximal value $a_6 \approx 20,25$.
(see example 2 above: the existence of an upper bound can only be shown if you know that (a_n) **decreases**.

Exercises on sequences and series No. 7

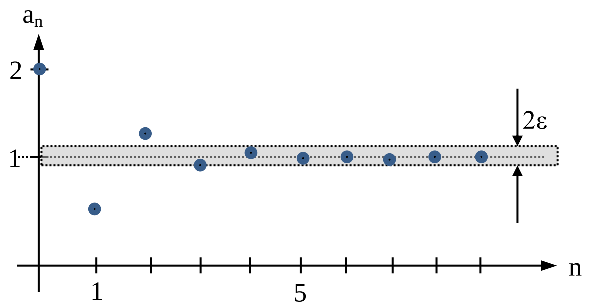
5.7.3. Limit of a sequence for n → ∞ (see also 5.1.1.)

Definition
A sequence (a_n) **converges** or **tends** to the **limit** a, if for any (especially small) $\varepsilon > 0$ existsts a n_ε, so that all terms after n_ε deviate by less than ε from this limit: $|a_n - a| < \varepsilon$ for $n \geq n_\varepsilon$. The notation is $a_n \to a$ für $n \to \infty$ or $\lim\limits_{n\to\infty} a_n = a$

Example
Find the number n_ε so that all terms of the sequence $(a_n) = (-2)^{-n} + 1$ deviate by less than ε = 0,1 resp. 0,01 resp. 0,001 from the limit a = 1?
Solution:

$$|a_n - a| < \varepsilon$$
$$|(-2)^{-n}| < \varepsilon$$
$$2^{-n} < \varepsilon$$
$$-n \ln(2) < \ln(\varepsilon)$$
$$n > -\frac{\ln(\varepsilon)}{\ln(2)}$$

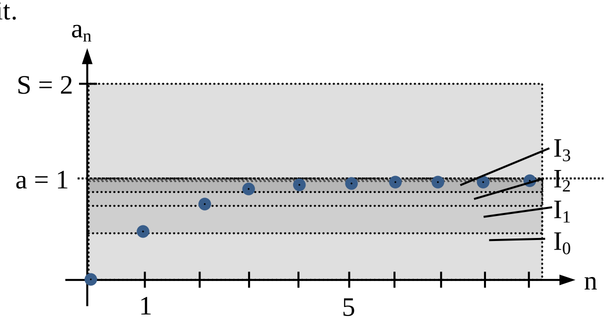
$\Rightarrow n_{0,1} = 4$, $n_{0,01} = 7$ and $n_{0,001} = 10$



Exercises on sequences and series No. 8

Theorem: If a sequence is monotonic and bounded it tends to a limit.

Proof:
Assume (a_n) is monotonically increasing and has the upper limit S.
Then we can find a least upper bound s with the following decreasing sequence of intervals (I_n):
We start with $I_0 = [a_0;S]$ and choose the next interval recursively as follows : I_k = upper half of I_{k-1}, if this upper half contains terms of (a_n) and I_k = lower half of I_{k-1}, if the upper half is empty.



Because every interval I_k is only half as large as and included in the previous interval I_{k-1} the upper and lower bounds of this telescoping chain (I_k) of intervals converge to the same limit s. s is an upper bound of (a_n), because all upper bounds of I_k are an upper bound of (a_n). s is also the least upper bound because each interval I_n contains elements of (a_n). s is also the limit of (a_n) for $n \to \infty$ because each interval I_k contains s and all elements of (a_n) beyond a certain n(k). But since the diameter of I_k is $S \cdot 2^{-k}$ the deviation $|s - a_n| < S \cdot 2^{-k}$ for all n > n(k) and for any given ε you can find a k with $S \cdot 2^{-k} < \varepsilon$.

Example:

Show that the sequence $a_n = \frac{n-2}{2n+1}$ is monotonic and bounded and therefore must converge.

Solution:

Monotonicity with the **ratio criterion**:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)-2}{2(n+1)+1} \cdot \frac{2n+1}{n-2}$$

$$= \frac{(n-1)(2n+1)}{(n-2)(2n+3)} \ ??$$

→ Better try **difference criterion**:

$$a_{n+1} - a_n = \frac{(n+1)-2}{2(n+1)+1} - \frac{n-2}{2n+1}$$

$$= \frac{n-1}{2n+3} - \frac{n-2}{2n+1}$$

$$= \frac{(n-1)(2n+1)-(n-2)(2n+3)}{(2n+1)(2n+3)}$$

$$= \frac{2n^2-n-1-(2n^2-n-6)}{(2n+1)(2n+3)}$$

$$= \frac{5}{(2n+1)(2n+3)}$$

$$> 0 \text{ for all } n \in \mathbb{N}$$

⇒ a_n is (strictly) monotonically increasing.

An **upper boundary** is for example S = 1, because

$$a_n < 1 \quad | \text{ plug in}$$

$$\frac{n-2}{2n+1} < 1 \quad | \cdot(2n+1) > 0$$

$$n-2 < 2n+1 \quad | -n \ ; -1$$

$$-3 < n$$

which holds for all $n \in \mathbb{N}$.

Since (a_n) is increasing and has an upper bound, it must converge to a limit.

This **limit** can be found easily by **polynomial long division** as in rational functions

$$\lim_{n\to\infty} \frac{n-2}{2n+1} = \lim_{n\to\infty}\left(\frac{1}{2} - \underbrace{\frac{5}{4n+2}}_{\to 0 \text{ for } n \to \infty}\right) = \frac{1}{2}.$$

Exercises on sequences and series No. 9

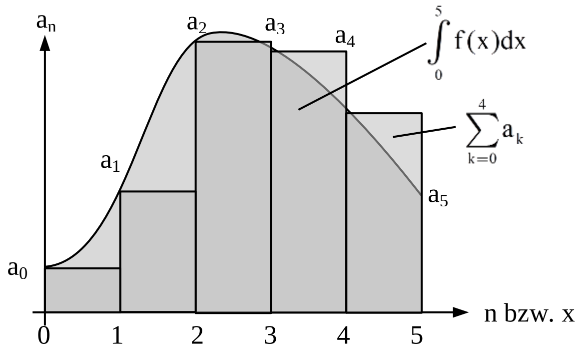
5.7.4. Series

Sigma notation and series

The corresponding **series** of a given sequence (a_n) is the sequence of the **sums of the first n terms** $\sum_{k=0}^{n} a_k := a_0 + a_1 + \ldots + a_n$,.

The letter **Sigma** Σ symbolizes the **sum** as does the S in an **integral.**

The **summation** of a sequence (a_n) to form a series $\sum_{k=0}^{n} a_k$ corresponds to the integration of a function f(x) to its **Integral** $\int_0^{n+1} f(x)dx$.



The **limit** of a **converging series** is denoted $\lim_{n\to\infty} \sum_{k=0}^{n} a_k = \sum_{k=0}^{\infty} a_k$.

Likewise the **limit** of a **converging integral** is denoted $\lim_{n\to\infty} \int_0^{n} f(x)dx = \int_0^{\infty} f(x)dx$ and called **improper integral.**

(The use of infinity ∞ as an upper bound actually is a contradiction and therefore an **improper** use of notation!)

Exercises on sequences and series No. 10

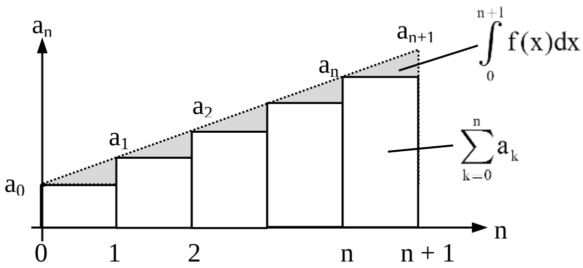
The arithmetic series

The summation of the arithmetic sequence $a_n = a_0 + d \cdot n$ results in the **arithmetic series**

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n$$
$$= a_0 + (a_0 + 1 \cdot d) + (a_0 + 2 \cdot d) + \ldots + (a_0 + (n-1) \cdot d) + (a_0 + n \cdot d)$$
$$= (n+1) \cdot a_0 + 1 \cdot d + 2 \cdot d + \ldots + (n-1) \cdot d + n \cdot d$$
$$= (n+1) \cdot a_0 + \underbrace{(n+1) \cdot d + (n+1) \cdot d + \ldots}$$
$$\begin{cases} \frac{1}{2} n \cdot (n+1) \cdot d, \text{ if n even} \\ \frac{1}{2}(n-1)(n+1)d + \frac{1}{2}(n+1) \cdot d = \frac{1}{2} n \cdot (n+1) \cdot d, \text{ if n odd} \end{cases}$$
$$= (n+1) \cdot a_0 + \frac{1}{2} n \cdot (n+1)d$$

Integrating the corresponding **linear function** $f(x) = a_0 + dx$ leads to the integral

$$\int_0^{n+1} f(x)dx = \left[a_0 x + \frac{1}{2} dx^2\right]_0^{n+1}$$
$$= (n+1)a_0 + \frac{1}{2}(n+1)^2 d.$$



The integral is larger than the corresponding sum, since the function is **monotonically increasing** and passes above the rectangles of the sum.

Example:

Determine the initial value a_0 and the common difference d of the arithmetic sequence (a_n) with $\sum_{k=0}^{4} a_k = 12$ and $\sum_{k=0}^{9} a_k = 29$.

Solution:

With $\sum_{k=0}^{4} a_k = 5a_0 + 10d = 12 \Leftrightarrow 10a_0 + 20d = 24$

and $\sum_{k=0}^{9} a_k = 10a_0 + 45d = 29$

subtracting the two equations results in $25d = 5 \Rightarrow$ common difference $d = \frac{1}{5}$ and initial value $a_0 = \frac{12}{5} - 2d = 2$

Exercises on sequences and series No. 11

The geometric series

The summation of the geometric sequence $a_n = a_0 \cdot q^n$ results in the **geometric series**

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

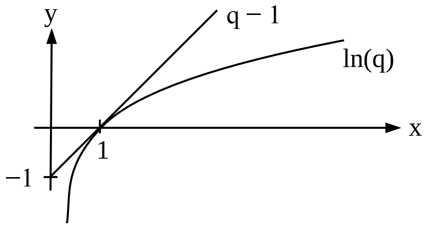
$$= a_0 + q \cdot a_0 + q^2 \cdot a_0 + \ldots + q^n \cdot a_0$$

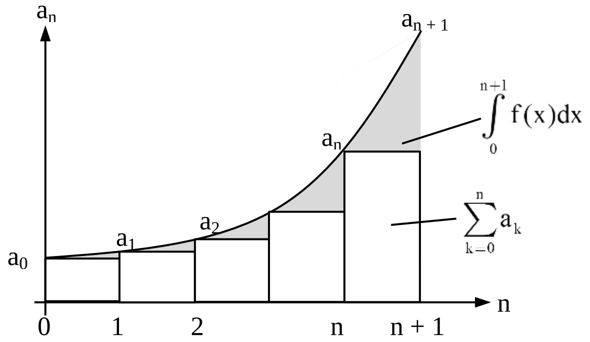
$$= a_0(1 + q + q^2 + \ldots + q^n)$$

$$= a_0 \cdot \frac{q^{n+1} - 1}{q - 1}, \text{ since}$$

$$(1 + q + q^2 + \ldots + q^n) \cdot (q - 1) = q + q^2 + q^3 + \ldots + q^{n+1} - (1 + q + q^2 + \ldots + q^n)$$

$$= q^{n+1} - 1$$





If counting starts with n = 1 we have

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

$$= a_1 + q \cdot a_1 + q^2 \cdot a_1 + \ldots + q^{n-1} \cdot a_1$$

$$= a_1(1 + q + q^2 + \ldots + q^{n-1})$$

$$= a_1 \cdot \frac{q^n - 1}{q - 1}$$

The integration of the corresponding **exponential function** $f(x) = a_0 \cdot q^x$ results in the integral

$$\int_0^{n+1} f(x)dx = \left[\frac{a_0 \cdot q^x}{\ln(q)}\right]_0^{n+1} = a_0 \cdot \frac{q^{n+1} - 1}{\ln(q)}$$

Becaus $q - 1 > \ln(q)$ (see drawing above right) the integral is slightly larger than the corresponding sum, since the function is **monotonically increasing** and passes above the rectangles of the sum.

Convergence of geometric series

For **decreasing** sequences with **negative percentages** $p < 0$ resp. common ratios $q < 1$ the power q^n vanishes for $n \to \infty$ and the geometric series **converges** to $\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = \lim_{n\to\infty} a_0 \cdot \frac{q^{n+1} - 1}{q - 1} = a_0 \cdot \frac{-1}{q - 1} = \frac{a_0}{1 - q}$

Example:

Determine the initial value a_0 and the common ratio d oft the geometric sequence a_n with $\sum_{k=0}^{2} a_k = 775$ and $\sum_{k=0}^{5} a_k = 781{,}2$.

Solution:

With $\sum_{k=0}^{2} a_k = a_0 \cdot \frac{q^3 - 1}{q - 1} = 775$ and $\sum_{k=0}^{5} a_k = a_0 \cdot \frac{q^6 - 1}{q - 1} = 781{,}2$ **dividing** the two equations results in $a_0 \cdot \frac{q^6 - 1}{q^3 - 1} = \frac{781{,}2}{775} \Leftrightarrow$ $\frac{(q^3 - 1)(q^3 + 1)}{q^3 - 1} = 1{,}008 \Rightarrow q^3 + 1 = 1{,}008 \Leftrightarrow q^3 = 0{,}008 \Rightarrow$ common ratio $q = 0{,}2$ and initial value $a_0 = 625$.

Exercises on sequences and series No. 12

integral criterion for the boundedness of series

Usually with the help of the various integration rules integrals are much easier to calculate than the corresponding sums. In many cases a good **upper bound** can be found by integrating a (possibly slightly adapted) corresponding function in suitable bounds. The function and the bounds have to be chosen so that the function is **always above the rectangles of the sum** inside the bounds. Likewise a **lower bound** may be found by integration a suitable function passing **inside the rectangles of the sum**

Example for the use of the integral criterion

Show that the series $\sum_{k=1}^{n} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3}$ is increasing and bounded above and therefore must converge.

Solution

Let $a_n = \frac{1}{n^3}$ be the generating sequence and $s_n = \sum_{k=1}^{n} a_k$ the corresponding series.

The series s_n is **monotonically increasing**, since $s_{n+1} - s_n = a_{n+1} = \frac{1}{(n+1)^3} > 0$ for all $n \in \mathbb{N}$

The generating sequence a_n is **monotonically decreasing**, since $a_{n+1} - a_n = \frac{1}{(n+1)^3} - \frac{1}{n^3} = \frac{n^3 - (n+1)^3}{(n+1)^3 \cdot n^3} < 0$ for all $n \in \mathbb{N}$.

s_n is **bounded above**, since

$s_n = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + ... + \frac{1}{n^3}$

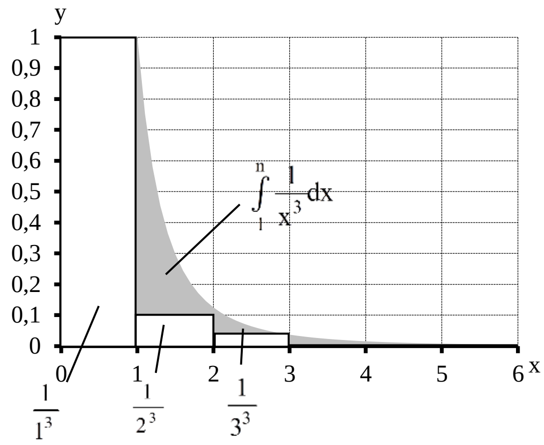
$< 1 + \int_1^n \frac{1}{x^3} dx$ (**integral criterion**, attention: adapted bounds!)

$= 1 + \left[-\frac{2}{x^2} \right]_1^n$

$= 1 - \frac{2}{n^2} + 2$

$= 3 - \frac{2}{n^2}$

< 3 for all $n \in \mathbb{N}$.



Therefore s_n must have a limit $\lim_{n \to \infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k^3} < 3$.

However the exact value of this limit is still unknown! **Leonhard Euler** showed in 1736 that

$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + ... = \frac{\pi^2}{6}$ and $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + ... = \frac{\pi^2}{8}$

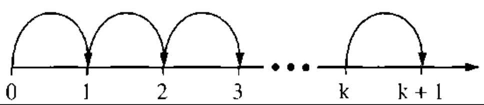
Exercises on sequences and series No. 13

5.7.5. Mathematical induction

Proof by mathematical induction (recursive proof)

In order to prove a statement for all $n \in \mathbb{N}$ it suffices to show

1. that the statement holds for n = 0 (**base case, starting point**)
2. that if the statement hold for n then it also holds for n + 1. (**inductive step, chain reaction**)



Example: (summation formula for the first n natural numbers)

Theorem: For all $n \in \mathbb{N}$ we have $1 + 2 + 3 + ... + n = \frac{1}{2} n(n + 1)$

Proof:

base case n = 1: $1 = \cdot \frac{1}{2} \cdot 1 \cdot (1 + 1) = 1$

inductive step $n \Rightarrow n + 1$:

inductive hypothesis for some n: $1 + 2 + 3 + ... + n = \frac{1}{2} n(n + 1)$

we have to show the statement for n + 1: $1 + 2 + 3 + ... + n + n + 1 = \frac{1}{2}(n + 1)(n + 2)$

Plug the inductive hypothesis into the left side: $1 + 2 + 3 + ... + n + n + 1 = \frac{1}{2} n(n + 1) + n + 1 = \frac{1}{2} n^2 + \frac{3}{2} n + 1$

right side: $\frac{1}{2}(n + 1)(n + 2) = \frac{1}{2} n^2 + \frac{3}{2} n + 1$ = **left side, qed**

Exercises on sequences and series Nos. 14 und 15