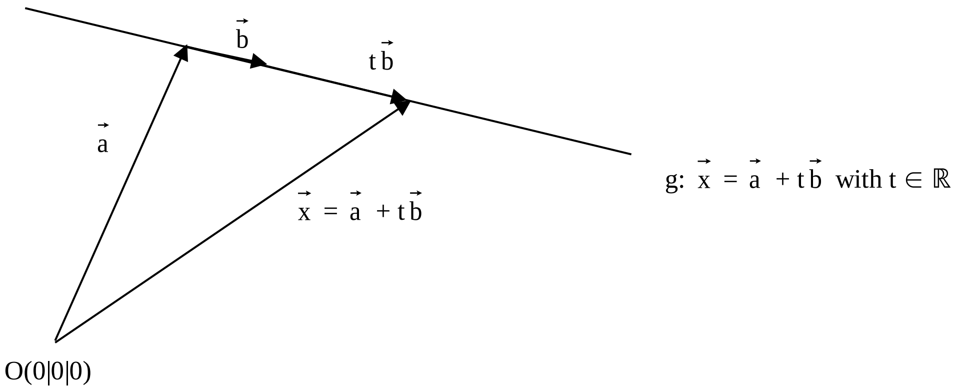
7.2. Lines in space

7.2.1. Vector equation of a line

Definition:

The **vector equation** g: $\vec{x}(t) = \vec{a} + t\vec{b}$ with $t \in \mathbb{R}$ gives the **position vector** $\vec{x}$ of a particle on the **line g** depending on the **parameter (time) t** with the **direction vector (velocity)** $\vec{b}$ and the **support vector (position vector of the starting point)** $\vec{a} = \vec{x}(0)$. The **speed** of the particle is the length $|\vec{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}$.

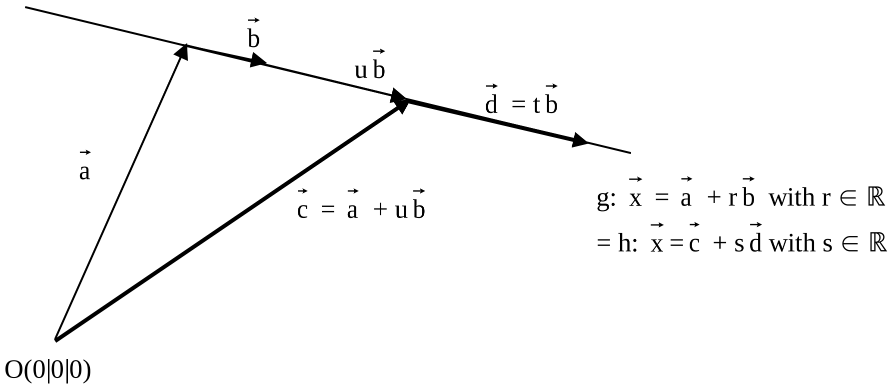


Theorem: (Change of direction vector and support vector)

Two given lines g: $\vec{x} = \vec{a} + r\vec{b}$ and h: $\vec{x} = \vec{c} + s\vec{d}$ with $r, s \in \mathbb{R}$ are equal, if there exist

$t \in \mathbb{R}$ with $\vec{d} = t\vec{b}$, i.e. the new direction vector $\vec{d}$ **is a multiple** of $\vec{b}$ (change of speed)

$u \in \mathbb{R}$ with $\vec{c} = \vec{a} + u\vec{b}$, i.e. the new support vector $\vec{c}$ **also lies on g**. (change of starting point)



Proof:

Each $\vec{x} \in h$ lies on g, since $\vec{x} = \vec{c} + s\vec{d} = \vec{a} + u\vec{b} + st\vec{b} = \vec{a} + (u + st)\vec{b} = \vec{a} + r\vec{b}$ with $r = u + st$

Each $\vec{x} \in g$ lies on h, since $\vec{x} = \vec{a} + r\vec{b} = \vec{c} - \frac{u}{t}\vec{d} + \frac{r}{t}\vec{d} = \vec{c} + (\frac{r}{t} - \frac{u}{t})\vec{d} = \vec{c} + s\vec{d}$ with $s = \frac{r}{t} - \frac{u}{t}$.

Exercises on lines Nos. 1 and 2

7.2.2. Lines through given points and with given directions

Theorem: **Lines through given points and in given directions**

- The line g in direction of the vector $\vec{a}$ through the point P has the equation g: $\vec{x} = \overrightarrow{OP} + r\,\vec{a}$
- The line g through the points P and Q has the equation g: $\vec{x} = \overrightarrow{OP} + r\,\overrightarrow{PQ}$

Sometimes vectors are given as **linear combination** of **basis vectors** $\vec{i} = \vec{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\vec{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Example: $\begin{pmatrix}2\\-3\\1\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} - 3\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} = 3\,\vec{i} - 3\,\vec{j} + \vec{k}$

Exercises on lines Nos. 3 - 6

7.2.3. Relations between lines

Theorem: **Relations between lines**

The two lines g: $\vec{x} = \vec{a} + r\,\vec{b}$ and h: $\vec{x} = \vec{c} + s\,\vec{d}$ have

- **no** common point if the equation $\vec{a} + r\,\vec{b} = \vec{c} + s\,\vec{d}$ has **no** solution.

  They are **parallel** if the equation $\vec{b} = t\,\vec{d}$ has **a** solution.

  skew, if the equation $\vec{b} = t\,\vec{d}$ has **no** solution
- one **intersection point** P if the equation $\vec{a} + r\,\vec{b} = \vec{c} + s\,\vec{d}$ has a **single** solution (r|s). The coordinates of P result when r and s are substituted in the corresponding equations for g and h.
- are **identical** if the equation $\vec{a} + r\,\vec{b} = \vec{c} + s\,\vec{d}$ has **many** solutions (r|s).

The **angle** between g and h can be obtained using the **scalar product** by $\alpha = \arccos\left(\frac{\vec{b} * \vec{d}}{|\vec{b}| * |\vec{d}|}\right)$

Exercises on lines Nos. 7 - 12

7.2.4. Cartesian equation of a line

Theorem: **Cartesian equation of a line**

The line g: $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} + t\cdot\begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix}$ can be written in the form $t = \frac{x-a_x}{b_x} = \frac{x-a_y}{b_y} = \frac{x-a_y}{b_y}$ if $b_x, b_y, b_z \neq 0$.

If $b_x = 0$ it has the form $x = a_x$ and $t = \frac{x-a_y}{b_y} = \frac{x-a_y}{b_y}$ and likewise for the other coordinates.

Exercises on lines Nos. 13 and 14