7.3. Exercises on planes

Exercise 1: Vector equation of a plane
Check whether P and Q are on E.

a) P(1|−1|1), Q(1|1|1), E_1: $\vec{x} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + r\begin{pmatrix}1\\-2\\1\end{pmatrix} + s\begin{pmatrix}-3\\1\\2\end{pmatrix}$

b) P(2|2|0), Q(3|2|0), E_2: $\vec{x} = \begin{pmatrix}2\\2\\1\end{pmatrix} + r\begin{pmatrix}-2\\-1\\1\end{pmatrix} + s\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

c) P(2|−3|0), Q(3|3|0), E_3: $\vec{x} = \begin{pmatrix}2\\-3\\0\end{pmatrix} + r\begin{pmatrix}2\\1\\3\end{pmatrix} + s\begin{pmatrix}0\\1\\1\end{pmatrix}$

d) P(0|2|1), Q(0|2|−1), E_4: $\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + r\begin{pmatrix}2\\1\\1\end{pmatrix} + s\begin{pmatrix}0\\-1\\1\end{pmatrix}$

Exercise 2: Intersects with coordinate axes
Find the intersects S_1, S_2 and S_3 with the coordinate axes of the planes E_1 – E_4 from exercise 1 and draw them into a three-dimensional coordinate system.

Exercise 3: Replacing support and direction vectors
Given are the planes E_1 – E_4 from exercises 1. Find new support vectors with starting points on a coordinate axis and new direction vectors parallel to the coordinate planes.

Exercise 4: Planes through given points
The planes E_1 - E_4 from exercise 1 constitute the side faces of a **pyramid** which is one half of an **octahedron**. Find four additional planes E_5 - E_8 to complete the octahedron. Choose **support vectors** parallel to the **coordinate axes** and **direction vectors** parallel to the **coordinate planes**.

Exercise 5: Plane through given points
Find the equation of the plane through the points A, B and C.

a) A(3|0|3), B(5|1|5), C(7|−1|0)

b) A(4|2|1), B(8|2|−2), C(2|5|4)

Exercise 6: Plane through two points with given direction
Find the plane through A and B parallel to g. Check whether g is on E.

a) A(2|−1|3), B(−1|2|2), g: $\vec{x} = \begin{pmatrix}3\\1\\3\end{pmatrix} + r\begin{pmatrix}4\\0\\-1\end{pmatrix}$

b) A(3|1|−1), B(2|1|0), g: $\vec{x} = \begin{pmatrix}-1\\0\\3\end{pmatrix} + r\begin{pmatrix}1\\2\\-3\end{pmatrix}$

Exercise 7: Plane through one point with two given directions
Find the plane through A parallel to g and h. Check whether g or h are on E.

a) $A\begin{pmatrix}2\\1\\3\end{pmatrix}$, g: $\vec{x} = \begin{pmatrix}3\\1\\3\end{pmatrix} + r\begin{pmatrix}4\\0\\-1\end{pmatrix}$, h: $\vec{x} = \begin{pmatrix}0\\1\\2\end{pmatrix} + r\begin{pmatrix}1\\1\\-1\end{pmatrix}$

b) $A\begin{pmatrix}1\\-1\\4\end{pmatrix}$, g: $\vec{x} = \begin{pmatrix}2\\1\\4\end{pmatrix} + r\begin{pmatrix}3\\1\\1\end{pmatrix}$, h: $\vec{x} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + r\begin{pmatrix}1\\0\\-1\end{pmatrix}$

Exercise 8: Plane and line
Examine E and g with regard to common points.

a) g: $\vec{x} = \begin{pmatrix}8\\5\\5\end{pmatrix} + r\begin{pmatrix}3\\-1\\5\end{pmatrix}$, E: $\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + r\begin{pmatrix}4\\8\\-4\end{pmatrix} + s\begin{pmatrix}6\\3\\3\end{pmatrix}$

b) g: $\vec{x} = \begin{pmatrix}3\\-1\\4\end{pmatrix} + r\begin{pmatrix}-1\\1\\2\end{pmatrix}$, E: $\vec{x} = \begin{pmatrix}9\\3\\-1\end{pmatrix} + r\begin{pmatrix}8\\-2\\2\end{pmatrix} + s\begin{pmatrix}-2\\2\\1\end{pmatrix}$

c) g: $\vec{x} = \begin{pmatrix}3\\1\\4\end{pmatrix} + r\begin{pmatrix}1\\2\\1\end{pmatrix}$, E: $\vec{x} = \begin{pmatrix}-2\\1\\-3\end{pmatrix} + r\begin{pmatrix}2\\0\\1\end{pmatrix} + s\begin{pmatrix}3\\-2\\1\end{pmatrix}$

d) g: $\vec{x} = \begin{pmatrix}0\\10\\-7\end{pmatrix} + r\begin{pmatrix}-1\\-8\\13\end{pmatrix}$, E: $\vec{x} = \begin{pmatrix}0\\2\\3\end{pmatrix} + r\begin{pmatrix}1\\-4\\2\end{pmatrix} + s\begin{pmatrix}-4\\0\\12\end{pmatrix}$

Exercise 9: Two planes
Examine E_1 and E_2 with regard to common points.

a) $E_1: \vec{x} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + r\begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix}$ $E_2: \vec{x} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} + r\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

b) $E_1: \vec{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $E_2: \vec{x} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

c) $E_1: \vec{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + r\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $E_2: \vec{x} = \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix} + r\begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$

d) $E_1: \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + r\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $E_2: \vec{x} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + r\begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} + s\begin{pmatrix} 6 \\ -5 \\ 3 \end{pmatrix}$

Exercise 10: Translation vector equation → coordinate equation
Find the coordinate equation of E:

a) E: $\vec{x} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ b) E: $\vec{x} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\cdot\begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$ c) E: $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s\cdot\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$

Exercise 11: Translation coordinate equation → vector equation
Find the vector equation of E:

a) E: $2x_1 + 3x_2 + 6x_3 = 24$ b) E: $x_1 - x_2 + 2x_3 = 3$ c) E: $4x_1 - 2x_2 - x_3 = 1$

Exercise 12: Intersects with coordinate axes and trace lines
Find the intersects with the coordinate axes and the intersects lines with the coordinate planes. Draw the corresponding triangular section of E into a coordinate system.

a) E: $2x_1 + 3x_2 + 6x_3 = 12$ b) E: $x_1 - 2x_2 + 2x_3 = 4$ c) E: $4x_1 - 2x_2 - x_3 = 8$ d) E: $-2x_1 - x_2 - 3x_3 = 6$

e) E: $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + s\cdot\begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$

Exercise 13: Common points of two planes
Find the common points of E and F

a) E: $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ und F: $4x_1 - 2x_2 + 3x_3 = 11$ b) E: $x_3 = 1$ und F: $\vec{x} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s\cdot\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

c) E: $x_1 - x_2 + x_3 = 2$ und F: $2x_1 + 2x_2 + 3x_3 = 5$ d) E: $x_1 + x_2 + x_3 = 1$ und F: $-x_1 + 2x_2 - 3x_3 = 2$

Exercise 14: Common points of two planes
Explain with geometric arguments why a system of two equations with three variables can have no or many solutions but never a single one.

Exercise 15: Relation between two planes
1. Show that E_1 and E_2 are parallel.
2. Find the coordinate form and all intersects with the coordinate axes..

a) E: $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, F: $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + s\cdot\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ b) E: $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r\cdot\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + s\cdot\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, F: $\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\cdot\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Exercise 16: Common points of a plane and a line
Explain with geometric arguments why a system of three equations with three variables can have either no or exactly one or infinitely many solutions but never e.g. two.

Exercise 17: Common points of planes and lines

1. Find the intersects with the coordinate axes of the plane E.
2. Find the intersects with the coordinate planes of the line g.
3. Determine $E \cap g$ and draw E and g in a common coordinate system.

a) E: $x_1 + x_2 + 2x_3 = 4$ and g: $\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ b) E: $2x_1 + 3x_2 + x_3 = -6$ and g: $\vec{x} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

c) E: $-3x_1 + 4x_2 + 6x_3 = 12$ und g: $\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Exercise 18: Cutting angle between plane and line

Calculate the angles between the line g: $\vec{x} = t\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ and the three coordinate axes as well as the three coordinate planes.

Exercise 19: Cutting angle between plane and line

Given are the plane E through A(1|0|1), B(2|1|1) and C(1|2|5) and the line g through G(4|−1|2) and H(1|1|3).

a) Find the intersection point of E and g.
b) Determine the angle between E and g.
c) The line g' is the reflection of g in E. Find an equation for g'.
d) Calculate the angle between g and g'.

Exercise 20: Cutting angle between two planes

Find the cutting angle between E and F

a) E: $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, F: $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ b) E: $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + r\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}$, F: $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Exercise 21: Pyramids

Given are the points A(1|−2|3), B(9|0|−1), C(8|2|−2) and E(6|3|5).

a) Find the point D so that the quadrangle ABCD is a parallelogram.
b) Determine the area of the quadrangle ABCD.
c) Show that E has equal distance to the four points A, B, C and D.
d) Calculate the volume of the pyramid ABCDE.
e) Find the angles α_1 and α_2 between the side faces and the base of the pyramid.
f) Find the angle β between the side edges and the base of the pyramid.

Exercise 22: Distance between a point and a plane

Find the distance of the point P from the plane E

a) P(14|6|25) and E: $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + s\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$ b) P(4|−5|4) and E: $\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Exercise 23: Distance between a line and a plane

Show that E: $7x_1 + 4x_2 - 4x_3 = 21$ is parallel to the line g through A(−11|−8|8) and B(−7|−1|22) and find their distance.

Exercise 24: Distance between two planes

Show that E: $x_1 - 2x_2 - 2x_3 = 2$ is parallel to F: $\vec{x} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and find their distance.

Exercise 25: Pyramid

Given are the points A(2|1|4), B(3|3|7), C(5|0|6) and D(7|4|3).

a) Show that the triangle ABC is equilateral and find its area.
b) Find the distance of the point D from the plane E through A, B and C.
c) Determine the volume of the pyramid ABCD.
d) Find the perpendicular F of the point D on the plane E and decide whether it is inside the base area ABC. **Hint**: Find the values of the parameters r and s in the equation $\overrightarrow{AF} = r\,\overrightarrow{AB} + s\,\overrightarrow{AC}$. Use a drawing to deduce the position of F in relation to A, B and C.

Exercise 26: Cone

Given are the points A(4|0|1), B(0|3|0), C(−2|1|3) and S(3|3|12).

a) Find the distance of the point S from the plane E through A, B and C.
b) Find the perpendicular F of the point S on the plane E.
c) Find the volume of the cone which is generated by rotation of the straight section [SC] around the axis [FS].

Exercise 27: Distance between a point and a line

Find the distance of the point P from the line g through A and B:

a) P(8|10|0), A(1|−4|0), B(3|2|3) b) P(−2|5,5|6,5), A(−2|−2|−1), B(4|4|5) c) P(6|8|10), A(−8|−4|8), B(6|3|−6)

Aufgabe 28: Distance between two lines

Find thew common normal unit vector and the distance between f and g.

a) g: $\vec{x} = \begin{pmatrix}1\\0\\1\end{pmatrix} + r\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and f: $\vec{x} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + r\begin{pmatrix}1\\2\\-1\end{pmatrix}$ b) g: $\vec{x} = \begin{pmatrix}8\\2\\1\end{pmatrix} + r\begin{pmatrix}2\\1\\0\end{pmatrix}$ and f: $\vec{x} = \begin{pmatrix}2\\5\\1\end{pmatrix} + r\begin{pmatrix}10\\5\\6\end{pmatrix}$

c) g: $\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + r\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and f: $\vec{x} = \begin{pmatrix}2\\3\\0\end{pmatrix} + r\begin{pmatrix}0\\-1\\-1\end{pmatrix}$

Exercise 29: Velocities, distance between lines

Three aircraft F_1, F_2 und F_3 start at time t = 0 in in the points $P_1(0|1|9)$, $P_2(7|2|-6)$ and. $P_3(2|0|1)$. They move with constant velocities $\vec{v}_1 = \begin{pmatrix}1\\3\\-1\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}-1\\2\\3\end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix}-3\\1\\2\end{pmatrix}$. Positions are in m and velocities in m/s.

a) Show that F_1 and F_2 move in a common plane E and give its coordinate form.
b) In which point S intersect the flight lines of F_1 and F_2 and when do they pass this point?
c) In which point T and in which angle α does the flight line of F_3 meet the plane E?
d) Find the distance of F_3 from E after two seconds.
e) After how many seconds is F_3 exactly in the middle between F_1 and F_2?
f) What is the minimal distance between F_1 and F_2?

Exercise 30: Velocities and Reflection of lines

The party jet of the sultan of Brunei is approaching Hong Kong airport. At time t = 0 it is 10 km to the south and 11 km to the east from the island Lantau at a height of 5 km.

The irish pilot has set the course on a straight line through a point with height of 3 km directly above Lantau. His speed is540 km/h and he is losing height at 20 metres per second. His whiskey is long gone.

The first and only aircraft of China Airsea Corp., a rickety and cheaply bought B737 at time t = 0 passes a point 4 km above the sea 20 km east and 4 km south of Lantau with 720 km/h heading due west. The chinese pilot's soothsayer has predicted impending disaster.

a) Draw the starting points and the flight paths of the two planes in a coordinate system with 1 : 100 000 (1 cm ≙ 1 km). Choose Lantau as origin with x_1-axis in southern direction, x_2- axis in eastern direction and x_3-axis pointing upwards. (3)
b) What is the minimal distance between the two flightpaths? (5)
c) What is the minimal distance between the two aircraft and when is it reached? (4)
d) At t = 0 s the sultan is contemplating his French built luxury yacht lying 2 km south of Lantau at anchor on the glittering sea. The sun stands 45° above the horizon due south. A sunray is reflected by the yacht's slanted window and blinds the sultan for a little moment. Find the slanting angle of the window to the vertical. (5)

7.3. Solutions to the exercises on planes

Exercise 1: vector equation of a plane

a) $P \in E_1$ with $\overrightarrow{OP} = \vec{a} + \frac{4}{5}\vec{b} - \frac{2}{5}\vec{c}$ and $Q \notin E_1$

b) $Q \in E_2$ with $\overrightarrow{OP} = \vec{a} - \frac{1}{3}\vec{b} + \frac{1}{3}\vec{c}$ and $P \notin E_2$

c) $P \in E_3$ with $\overrightarrow{OP} = \vec{a} - 3\vec{c}$ and $Q \notin E_3$

d) $Q \in E_4$ with $\overrightarrow{OP} = \vec{a} - \frac{1}{2}\vec{b} - \frac{3}{2}\vec{c}$ and $P \notin E_4$

Exercise 2: Intersection with coordinate axes

a) $S_1(1|0|0)$, $S_2(0|1|0)$, $S_3(0|0|1)$

b) $S_2(1|0|0)$, $S_2(0|-1|0)$, $S_3(0|0|1)$

c) $S_1(-1|0|0)$, $S_2(0|-1|0)$, $S_3(0|0|1)$

d) $S_1(-1|0|0)$, $S_2(0|1|0)$, $S_3(0|0|1)$

Exercise 3: Replacing support and direction vectors

a) $E_1: \vec{x} = \begin{pmatrix}0\\0\\1\end{pmatrix} + u\begin{pmatrix}1\\0\\-1\end{pmatrix} + v\begin{pmatrix}0\\1\\-1\end{pmatrix}$, (upper right front)

b) $E_2: \vec{x} = \begin{pmatrix}0\\0\\1\end{pmatrix} + u\begin{pmatrix}1\\0\\-1\end{pmatrix} + v\begin{pmatrix}0\\-1\\-1\end{pmatrix}$,(upper right back)

c) $E_3: \vec{x} = \begin{pmatrix}0\\0\\1\end{pmatrix} + u\begin{pmatrix}-1\\0\\-1\end{pmatrix} + v\begin{pmatrix}0\\-1\\-1\end{pmatrix}$,(upper left back)

d) $E_4: \vec{x} = \begin{pmatrix}0\\0\\1\end{pmatrix} + u\begin{pmatrix}-1\\0\\-1\end{pmatrix} + v\begin{pmatrix}0\\1\\-1\end{pmatrix}$,(upper left front)

Exercise 4: Planes through given points

a) $E_5: \vec{x} = \begin{pmatrix}0\\0\\-1\end{pmatrix} + u\begin{pmatrix}1\\0\\1\end{pmatrix} + v\begin{pmatrix}0\\1\\1\end{pmatrix}$, (lower right front)

b) $E_6: \vec{x} = \begin{pmatrix}0\\0\\-1\end{pmatrix} + u\begin{pmatrix}1\\0\\1\end{pmatrix} + v\begin{pmatrix}0\\-1\\1\end{pmatrix}$, (lower right back)

c) $E_7: \vec{x} = \begin{pmatrix}0\\0\\-1\end{pmatrix} + u\begin{pmatrix}-1\\0\\1\end{pmatrix} + v\begin{pmatrix}0\\-1\\1\end{pmatrix}$, (lower left back)

d) $E_8: \vec{x} = \begin{pmatrix}0\\0\\-1\end{pmatrix} + u\begin{pmatrix}-1\\0\\1\end{pmatrix} + v\begin{pmatrix}0\\1\\1\end{pmatrix}$, (lower left front)

Exercise 5: Plane through three points

a) $E: \vec{x} = \begin{pmatrix}3\\0\\3\end{pmatrix} + r\begin{pmatrix}2\\1\\2\end{pmatrix} + s\begin{pmatrix}4\\-1\\-3\end{pmatrix}$,

b) $E: \vec{x} = \begin{pmatrix}4\\2\\1\end{pmatrix} + r\begin{pmatrix}4\\0\\-3\end{pmatrix} + s\begin{pmatrix}-2\\-3\\3\end{pmatrix}$

Exercise 6: Plane through two points with a given direction

a) $E: \vec{x} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + r\begin{pmatrix}-3\\3\\-1\end{pmatrix} + s\begin{pmatrix}4\\0\\-1\end{pmatrix}$,

b) $E: \vec{x} = \begin{pmatrix}3\\1\\-1\end{pmatrix} + r\begin{pmatrix}1\\0\\-1\end{pmatrix} + s\begin{pmatrix}1\\2\\-3\end{pmatrix}$

Exercise 7: Plane through one points with two given directions

a) $g, h \not\subset E: \vec{x} = \begin{pmatrix}2\\1\\3\end{pmatrix} + r\begin{pmatrix}4\\0\\-1\end{pmatrix} + s\begin{pmatrix}1\\1\\-1\end{pmatrix}$,

b) $g, h \not\subset E: \vec{x} = \begin{pmatrix}1\\-1\\4\end{pmatrix} + r\begin{pmatrix}3\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\0\\-1\end{pmatrix}$

Exercise 8: Plane and line

a) $g \cap E = S(5|6|0)$ b) $g \cap E = S(11|-9|-12)$ c) $g \cap E = \{\}$ with $g \,||\, E$ d) $g \subset E$

Exercise 9: Two planes

a) $E_1 \cap E_2 = g: \vec{x} = \begin{pmatrix}6\\0\\0\end{pmatrix} + r\begin{pmatrix}-2\\0\\1\end{pmatrix}$, b) $E_1 \cap E_2 = g: \vec{x} = \begin{pmatrix}6\\0\\1\end{pmatrix} + r\begin{pmatrix}-1\\1\\0\end{pmatrix}$, c) $E_1 = E_2$ d) $E_1 \cap E_2 = \{\}$ with $E_1||E_2$.

Exercise 10: Translation vector equation → coordinate equation

a) $-x_1 + x_2 + x_3 = 4$ b) $x_1 - 2x_2 + x_3 = -6$ c) $2x_1 + 5x_2 - x_3 = 10$

Exercise 11: Translation coordinate equation → vector equation

a) E: $\vec{x} = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix} + r\cdot\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + s\cdot\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ b) E: $\vec{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + r\cdot\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + s\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ c) E: $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s\cdot\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

Exercise 12: Intersects with the coordinate axes

a) $S_{12}(0|0|-2)$, $S_{13}(0|-4|0)$ and $S_{23}(-6|0|0)$ b) $S_{12}(0|0|-2)$, $S_{13}(0|2|0)$ and $S_{23}(4|0|0)$

c) $S_{12}(0|0|-8)$, $S_{13}(0|-4|0)$ and $S_{23}(-2|0|0)$ d) $S_{12}(0|0|-2)$, $S_{13}(0|-6|0)$ and $S_{23}(-3|0|0)$

e) E: $6x_1 + 4x_2 + 3x_3 = 12 \Rightarrow S_{12}(0|0|4)$, $S_{13}(0|3|0)$ and $S_{23}(2|0|0)$

Exercise 13: Intersecting lines of two planes

a), b) g: $\vec{x} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + r\cdot\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ c) g: $\vec{x} = \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix} + r\cdot\begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$ d) g: $\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + r\cdot\begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix}$

Exercise 14: Common points of two planes

The solutions of a system of two equations for three variables can be interpreted as the common points of the two planes represented by the two equations. They can be parallel (no solution), identical (many solutions depending on two parameters) or they can have an intersecting line (many solutions depending on one variable). Since it is geometrically impossible for two planes to meet only in one point the system cannot have a single solution.

Exercise 15: Relation of two planes

a) E: $x_2 = 1$ with $S_2(0|1|0)$ and F: $x_2 = 2$ with $S_2(0|2|0)$ have the same normal vector $\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b) E: $x_2 + x_3 = 2$ with $S_2(0|2|0)$, $S_3(0|0|2)$ and E: $x_2 + x_3 = 4$ with $S_2(0|4|0)$, $S_3(0|0|4)$ have the same normal vector $\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Exercise 16: Common points of two planes

The solutions of a system of three equations for three variables can be interpreted as the common points of the three planes represented by the three equations. They can be parallel (no solution), identical (many solutions depending on two parameters) or they can have a single intersecting line (many solutions depending on one variable). The usual case will be neither of the above. There will be two or three intersecting lines of different pairs of planes. Only if these intersecting lines meet in one point there is a single solution.

Exercise 17: Common points of planes and lines

a) E with $S_1(4|0|0)$, $S_2(0|4|0)$, $S_3(0|0|2)$ and g with $S_{23}(0|2|2)$, $S_{13}(-1|0|1)$, $S_{12}(-2|-2|0)$ meet in $S_{Eg}(-\frac{2}{5} | \frac{6}{5} | \frac{8}{5})$

b) E with $S_1(-3|0|0)$, $S_2(0|-2|0)$, $S_3(0|0|-6)$ and g with $S_{23}(0|4|-2)$, $S_{13}(-2|0|-4)$, $S_{12}(2|8|0)$ meet in $S_{Eg}(-\frac{50}{9} | \frac{4}{9} | -\frac{34}{9})$

c) E with $S_1(-4|0|0)$, $S_2(0|3|0)$, $S_3(0|0|2)$ and g with $S_{23}(0|1|0)$, $S_{13}(1|0|1)$, $S_{12}(0|1|0)$ meet in $S_{Eg}(-\frac{2}{39} | \frac{21}{13} | \frac{8}{13})$

Exercise 18: Cutting angle between plane and line

a) $\alpha_1 \approx 78{,}2°$, $\alpha_2 \approx 42{,}0°$, $\alpha_3 \approx 56{,}1°$ b) $\alpha_{12} \approx 48{,}0°$; $\alpha_{23} \approx 21{,}8°$, $\alpha_{13} \approx 33{,}8°$

Exercise 19: Cutting angle between plane and line

a) E: $2x_1 - 2x_2 + x_3 = 3$, g: $\vec{x} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + r\cdot\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ with $S(1|1|3)$ b) $\sin(\alpha) = \frac{9}{3\sqrt{14}} \Rightarrow \alpha \approx 53{,}3°$

c) g': $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + r\cdot\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ d) $\cos(\beta) = -\frac{2}{7} \Rightarrow \beta \approx 106{,}6° = 2\alpha$

Exercise 20: Cutting angle between two planes

a) E: $x_1 + 2x_3 = 3$ und F: $x_1 + x_2 = 1 \Rightarrow \alpha \approx 71{,}6°$

b) $E = F = -2x_1 + x_2 = 0 \Rightarrow \alpha \approx 0°$

Exercise 21: Pyramids

a) D(0|0|2), ABCD is even a rectangle.
b) $A = 6\sqrt{14}$
c) $V = 42$
d) $= 92{,}85$
e) $\alpha_1 = 77{,}69°$, $\alpha_2 \approx 50{,}77°$
f) $\beta \approx 49{,}80°$

Exercise 22: Distance between a point and a plane

a) E: $4x_1 - 3x_2 + 12x_3 = 0 \Rightarrow d = 26$

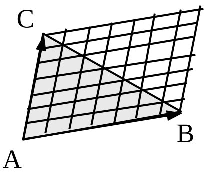
b) E: $x_1 + 2x_2 + 3x_3 = 0$ with $P \in E \Rightarrow d = 0$

Exercise 23: Distance between a line and a plane
d = 18 LE

Exercise 24: Distance between two planes
F: $x_1 - 2x_2 - 2x_3 = 5$ with d = 1

Exercise 25: Pyramid

a) $\overline{AB} = \overline{BC} = \overline{CA} = \sqrt{14} \Rightarrow A = \frac{7}{2}\sqrt{3}$



b) F: $\vec{x} = \begin{pmatrix}2\\1\\4\end{pmatrix} + r\begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}3\\-1\\2\end{pmatrix} \Rightarrow$ E: $-x_1 - x_2 + x_3 = 1 \Rightarrow d = 3\sqrt{3}$

c) V = 10,5

d) F(4|1|6) with $\overrightarrow{AF} = \frac{2}{7}\overrightarrow{AB} + \frac{4}{7}\overrightarrow{AC} \Rightarrow$ F is inside the triangle ABC, since $\frac{2}{7} + \frac{4}{7} < 1$ (see drawing)

Exercise 26: Cone

a) E: $x_1 + 2x_2 + 2x_3 = 6 \Rightarrow d = 9$

b) F(0|−3|6)

c) $r = \sqrt{29}$ LE $\Rightarrow V = 78\pi$

Exercise 27: Distance between a point and a line

a) Perpendicular F(5|8|6) $\Rightarrow d = 7$

b) Perpendicular F(3|3|4) $\Rightarrow d = 2{,}5\sqrt{6}$

c) Perpendicular F(0|0|0) $\Rightarrow d = 10\sqrt{2}$

Exercise 28: Distance between two skew lines

a) $\overrightarrow{n_0} = \frac{1}{\sqrt{50}}\begin{pmatrix}-3\\4\\5\end{pmatrix}$ and $d = \frac{6}{5}\sqrt{2}$

b) $\overrightarrow{n_0} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\\0\end{pmatrix}$ and $d = \frac{12}{5}\sqrt{5}$

c) $\overrightarrow{n_0} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $d = \frac{1}{\sqrt{3}}$

Exercise 29: Velocities, distance between two skew lines

a) E: $11x_1 - 2x_2 + 5x_3 = 43$
b) S(3|10|6) is passed by F_1 after 3 seconds and by F_2 after 4 seconds
c) $T(\frac{98}{25}|-\frac{16}{25}|-\frac{7}{25})$ und $\alpha \approx 33{,}06°$
d) $d = \frac{11}{5}\sqrt{6}$ m $\approx 5{,}39$ m
e) $d_{13} = d_{23}$ at t = 3,167 s
f) $d_{12min} = 2{,}67$ m is reached at $t = \frac{25}{7}$ s

Exercise 30: Velocities and reflection of lines

a) Drawing

b) Origin O(0|0|0) in Lantau, distances in m, time in s, speeds in m/s ⇒ 540 km/h = 150 m/s = v· $\sqrt{10^2+11^2+20^2}$ = v·15 ⇒ speed v = 10 ⇒ Party jet g: $\vec{x}(t) = \begin{pmatrix}10000\\11000\\5000\end{pmatrix} + t\cdot\begin{pmatrix}-100\\-110\\-20\end{pmatrix}$ and B 737 h: $\vec{y}(t) = \begin{pmatrix}4000\\20000\\4000\end{pmatrix} + t\cdot\begin{pmatrix}0\\200\\0\end{pmatrix}$ ⇒ common normal unit vector $\vec{n}_0 = \frac{1}{\sqrt{26}}\begin{pmatrix}1\\0\\-5\end{pmatrix}$ ⇒ distance d = $\left|\frac{1}{\sqrt{26}}\left[\begin{pmatrix}10000\\11000\\5000\end{pmatrix}-\begin{pmatrix}4000\\0\\4000\end{pmatrix}\right]*\begin{pmatrix}1\\0\\-5\end{pmatrix}\right| = \frac{1000}{\sqrt{26}} \approx 196{,}1$ m.

c) Distance $d_{12}(t) = \sqrt{\left(\vec{y}-\vec{x}\right)^2} = \sqrt{(6000-100\cdot t)^2+(9000-90\cdot t)^2+(1000-20\cdot t)^2}$ ⇒ (GTR) absolute min at t = 77,3 s with d_{12min} = 2732,2 m

d) Yacht lies in P(2000|0|0), Partyjet is in Q(10000|−11000|5000) ⇒ reflected sunray g': $\vec{y} = \begin{pmatrix}2000\\0\\0\end{pmatrix} + t'\cdot\begin{pmatrix}8\\-11\\5\end{pmatrix}$

Incoming sunray has angle of incidence α_1 = 45 ° and reflected sunray has $\alpha_2 = \cos^{-1}\left(\frac{\begin{pmatrix}8\\-11\\5\end{pmatrix}*\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{210}\cdot 1}\right) \approx \cos^{-1}(0{,}345) \approx 69{,}8°$ to vertical axis. Since angle of incidence equals angle of reflection the normal vector on the window pane has angle α = $\frac{\alpha_1+\alpha_2}{2}$ = 57,4° to vertical and the window pane itself is inclined 42,6° to vertical axis.