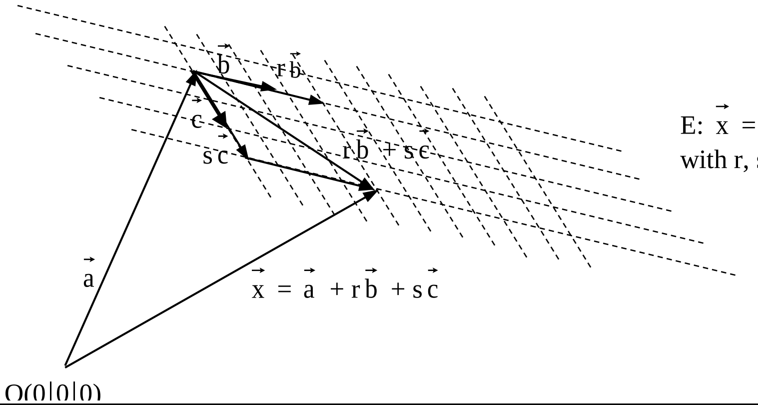
7.3. Planes

7.3.1. Vector equation of a plane

Definition:

A vector equation of the form E: $\vec{x} = \vec{a} + r\vec{b} + s\vec{c}$ with parameters r, s ∈ ℝ defines a plane with **support vector** $\vec{a}$ and **direction vectors** $\vec{b}$ and $\vec{c}$, **which must not be parallel** (!).



Exercises on planes Nos. 1 - 3

7.3.2. Planes through given points and with given directions

Theorem: Planes through given points with given directions

The plane E parallel to the vectors $\vec{b}$ und $\vec{c}$ through the point P is E: $\vec{x} = \overrightarrow{OP} + r\vec{b} + s\vec{c}$.

The plane parallel to the vector $\vec{c}$ through the points P and Q is E: $\vec{x} = \overrightarrow{OP} + r\overrightarrow{PQ} + s\vec{c}$.

The plane through the points P, Q and R is E: $\vec{x} = \overrightarrow{OP} + r\overrightarrow{PQ} + s\overrightarrow{PR}$.

Exercises on planes Nos.4 - 7

7.3.3. Common points of a plane and a line

Theorem: **Common points of a plane and a line**

The common points of a line g: $\vec{x} = \vec{a} + r\vec{b}$ and a plane E: $\vec{x} = \vec{c} + s\vec{d} + t\vec{e}$ result from the equation $\vec{a} + r\vec{b} = \vec{c} + s\vec{d} + t\vec{e}$.

- g ∩ E = {}: g and E have **no** common point, if the equation has **no** solution. In this case g runs **parallel** to E.
- g ∩ E = {P}: g and E have an **intersections point** P, if the equation has a **single** solution (r|s|t). Its coordinates result from substituting r resp. s and t in the equations for g resp. E.
- g ⊂ E: g **lies in** E, if the equation has **many** solutions (r|s|t).

Exercises on planes No. 8

7.3.4. Common points of two planes

Theorem: **Common points of two planes**

The common points of E: $\vec{x} = \vec{a} + r\vec{b} + s\vec{c}$ and F: $\vec{x} = \vec{d} + t\vec{e} + u\vec{f}$ result from the equation $\vec{a} + r\vec{b} + s\vec{c} = \vec{d} + t\vec{e} + u\vec{f}$.

- E ∩ F = {}: E and F have **no** common point if the equation has **no** solution. In this case they are **parallel** to each other.
- E ∩ F = {g}: E and F have an **intersecting line** g, if the solutions (r|s|t|u) of the equation depend on a **single** parameter. The vector equation of g results from substituting r uand s resp. t and u in the equations for E resp. F.
- E = F: E and F are **identical**, if the solutions (r|s|t|u) of the equation depend on **two** parameters.

Example:

Find common points of E_1: $\vec{x} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + r_1\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix} + s_1\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}$ and E_2: $\vec{x} = \begin{pmatrix}0\\1\\2\end{pmatrix} + r_2\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix} + s_2\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}$

Solution

$$E_1 = E_2 \Leftrightarrow \begin{array}{c} \begin{matrix} r_1 & s_1 & r_2 & s_2 \end{matrix} \\ \left(\begin{array}{cccc|c} 3 & 2 & -1 & -1 & -1 \\ -1 & 0 & 3 & 0 & 1 \\ 1 & 1 & -2 & 1 & 4 \end{array}\right)\end{array} \Leftrightarrow \begin{array}{c} \begin{matrix} r_1 & s_1 & r_2 & s_2 \end{matrix} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1,5 & -5 \\ 0 & 1 & 0 & 1,5 & 6,\overline{3} \\ 0 & 0 & 1 & -0,5 & -1,\overline{3} \end{array}\right)\end{array} \text{ with GDC.}$$

Choose parameter $s_2 \in \mathbb{R} \Rightarrow r_2 = -1,\overline{3} + 0,5s_2$ and substitute in E_2

$$\Rightarrow \vec{x} = \begin{pmatrix}0\\1\\2\end{pmatrix} + r_2\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix} + s_2\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} + (-1,\overline{3} + 0,5s_2)\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix} + s_2\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} - 1,\overline{3}\begin{pmatrix}1\\-3\\2\end{pmatrix} + s_2\cdot 0,5\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix} + s_2\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} =$$

$$\begin{pmatrix}-1,\overline{3}\\4\\-0,\overline{6}\end{pmatrix} + s_2\cdot\begin{pmatrix}1,5\\-1,5\\0\end{pmatrix} = \text{intersecting line } E_1 \cap E_2.$$

Exercises on planes No. 9

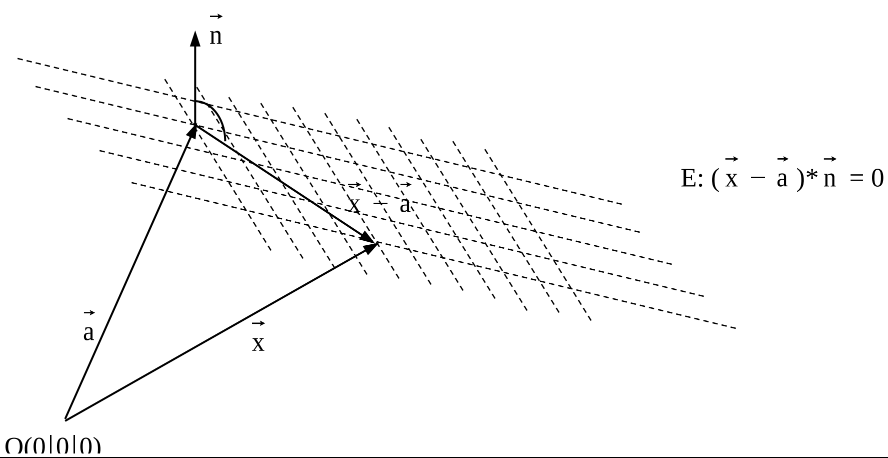
7.3.5. Coordinate equation of a plane

The direction of a plane can be given by a **single normal** (**perpendicular**) **vector** instead of the two direction vectors. The resulting coordinate equation is simpler than the vector equation and the normal vector is essential for calculating the **distances** between the plane and a point.

Definition: Coordinate equation of the plane

The plane perpendicular to the **normal vector** $\vec{n} = \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ through the **point A** with position vector $\vec{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ is the set of all points P with position vectors $\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ which comply with the equation E: $(\vec{x} - \vec{a})*\vec{n} = 0 \Leftrightarrow \vec{x}*\vec{n} = \vec{a}*\vec{n}$.

Writing down the coordinates results in the **coordinate form** E: $n_1x_1 + n_2x_2 + n_3x_3 = d$ with $d = \vec{a}*\vec{n}$.



Translation vector equation → coordinate equation:

E: $\vec{x} = \vec{a} + r\vec{b} + s\vec{c} = 0$ with $r, s \in \mathbb{R}$ has the **coordinate equation** E: $(\vec{x} - \vec{a})*\vec{n} = 0$ with **normal vector** $\vec{n} = \vec{b} \times \vec{c}$.

Example:

vector equation E: $\vec{x} = \vec{a} + r\cdot\vec{b} + s\cdot\vec{c} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + r\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix} + s\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}$

Normal vector $\vec{n} = \vec{b} \times \vec{c} = \begin{pmatrix}3\\-1\\1\end{pmatrix} \times\cdot \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}(-1)\cdot 1-0\cdot 1\\-3\cdot 1+2\cdot 1\\3\cdot 0-2\cdot(-1)\end{pmatrix} = \begin{pmatrix}-1\\-1\\2\end{pmatrix}$

coordinate equation E: $(\vec{x} - \vec{a})*\vec{n} = 0 \Leftrightarrow \vec{x}*\vec{n} = \vec{a}*\vec{n} \Leftrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}*\begin{pmatrix}-1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix}*\begin{pmatrix}-1\\-1\\2\end{pmatrix} \Leftrightarrow$ $-x - y + 2z = -5$

Exercises on planes No. 10

Translation coordinate equation → vector equation:

E: $(\vec{x} - \vec{a})*\vec{n} = 0$ has the vector equation E: $\vec{x} = \vec{a} + r\vec{b} + s\vec{c} = 0$ with r, s ∈ ℝ. Die **direction vectors** $\vec{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ and $\vec{c} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ must be perpendicular to $\vec{n} = \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ and must not be parallel to each other: $\vec{b} * \vec{n} = 0$ und $\vec{c} * \vec{n} = 0$ with $\vec{b} \neq t\cdot\vec{c}$.

Hence any two solutions of the system $\left|\begin{matrix}b_1n_1 + b_2n_2 + b_3n_3 & = & 0\\ c_1n_1 + c_2n_2 + c_3n_3 & = & 0\end{matrix}\right|$ with $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \neq \begin{pmatrix}tc_1\\tc_2\\tc_3\end{pmatrix}$ will provide possible $\vec{b}$ and $\vec{c}$

Example:
Find a vector equation to E: $x - 3y + 2z = 6$.

Solution:

Choose any Point P(x|y|z) which fits into equation, for example P(6|0|0) ⇒ $\vec{a} = \begin{pmatrix}6\\0\\0\end{pmatrix}$

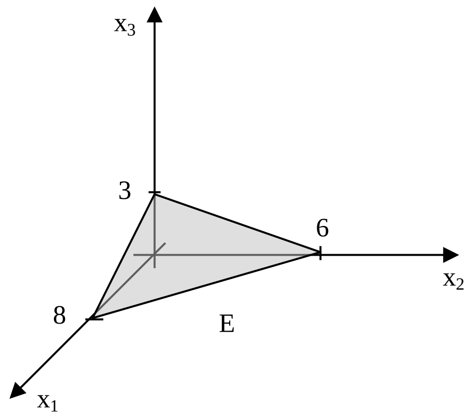
Choose any two direction vectors $\vec{b}$ and $\vec{c}$ perpendicular to normal vector $\vec{n} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$, i.e. with $\vec{b}*\vec{n} = \vec{c}*\vec{n} = 0$,

for example $\vec{b} = \begin{pmatrix}3\\1\\0\end{pmatrix}$ and $\vec{c} = \begin{pmatrix}0\\2\\3\end{pmatrix}$. Hence a possible vector equation is E: $\vec{x} = \begin{pmatrix}6\\0\\0\end{pmatrix} + r\begin{pmatrix}3\\1\\0\end{pmatrix} + s\begin{pmatrix}0\\2\\3\end{pmatrix}$.

Exercises on planes No. 11

Example: Graphical presentation of a plane using intersects with the coordinate axes
Given is the plane E: $3x_1 + 4x_2 + 8x_3 = 24$. Find the intersects with the coordinate axes and the intersect lines with the coordinate planes. Draw the corresponding triangular sector of E in a coordinate system.

Solution:
The intersects with the coordinate axes are obtained by setting zero the other two coordinates: E.g. on the x_1-axis we have $x_2 = x_3 = 0 \Rightarrow 3x_1 + 4\cdot 0 + 8\cdot 0 = 24 \Rightarrow x_1 = 8 \Rightarrow S_1(8|0|0)$. On the x_2- and x_3-axes we get $S_2(0|6|0)$ and $S_3(0|0|3)$.
The intersect lines with the coordinate planes are obtained by setting zero the remaining coordinate: E.g. on the x_1x_2-plane we have $x_3 = 0 \Rightarrow g_{12}: 3x_1 + 4x_2 = 24 \Leftrightarrow x_2 = 6 - \frac{3}{4}x_1$. On the x_1x_3-plane and the x_2x_3-plane we get $g_{13}: x_3 = 3 - \frac{3}{8}x_1$ and $g_{23}: x_3 = 3 - \frac{1}{2}x_2$.



Exercises on planes No. 12

7.3.6. Common points of two planes in coordinate equations

Relations between planes in coordinate equations
The common points $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of the planes $E_1: (\vec{x} - \vec{a}_1)*\vec{n}_1 = 0$ and $E_2\ (\vec{x} - \vec{a}_2)*\vec{n}_2 = 0$ are solutions of the system $\left| \begin{matrix} n_{11}x_1 + n_{12}x_2 + n_{13}x_3 & = & d_1 \\ n_{21}x_1 + n_{22}x_2 + n_{23}x_3 & = & d_2 \end{matrix} \right|$ with $d_1 = \vec{a}_1 * \vec{n}_1$ and $d_2 = \vec{a}_2 * \vec{n}_2$.

- $E_1 \cap E_2 = \{\}$: E_1 and E_2 have **no** common point if the system has **no** solution. In this case they are **parallel**.
- $E_1 \cap E_2 = \{g\}$: E_1 and E_2 have **an intersect line** g: $\vec{x} = \vec{a} + t\vec{b}$ if the system has many solutions $\vec{x} = \vec{a} + t\vec{b}$ depending on **one** parameter.
- $E_1 = E_2$: E_1 and E_2 are identical if the system has solutions $E_1 = E_2$: $\vec{x} = \vec{a} + t\vec{b} + u\vec{c}$ depending on **two** parameters.

Exercises on planes Nos. 13 - 15

7.3.7. Common points of three planes in coordinate equations

The three equations of a 3×3 system of linear equations can be interpreted as the common points of three planes:

Relations between three planes in coordinate equations
The common points of the three planes E_i: $(\vec{x} - \vec{a}_i)*\vec{n}_i = 0$ with $i \in \{1; 2; 3\}$ are solutions of the system $\left| \begin{matrix} n_{11}x_1 + n_{12}x_2 + n_{13}x_3 & = & d_1 \\ n_{21}x_1 + n_{22}x_2 + n_{23}x_3 & = & d_2 \\ n_{31}x_1 + n_{32}x_2 + n_{33}x_3 & = & d_3 \end{matrix} \right|$ with $d_i = \vec{a}_i * \vec{n}_i$. There are three cases:

- $E_1 \cap E_2 \cap E_3 = \{\}$: The three planes have **no common point** because
  - all three have the same direction and are **parallel** (Fig 1)
  - they meet pairwise in parallel intersecting lines which never meet. (Fig 2 and 3)
- $E_1 \cap E_2 \cap E_3 = \{P\}$: The three planes have one **intersection point** $P(x_1|x_2|x_3)$ if the system has a single solution $(x_1|x_2|x_3)$. (Fig 4)
- $E_1 \cap E_2 \cap E_3 = \{g\}$: The three planes have a **intersecting line** or **common axis** g: $\vec{x} = \vec{a} + t\vec{b}$ if the system has many solutions $\vec{x} = \vec{a} + t\vec{b}$ depending on a single parameter t. (Fig 5)

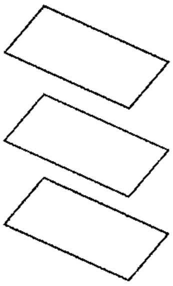


Fig 1

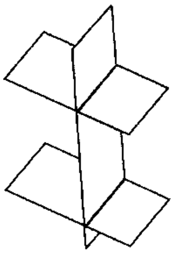


Fig 2

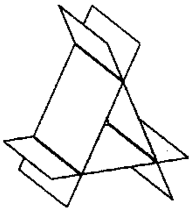


Fig 3

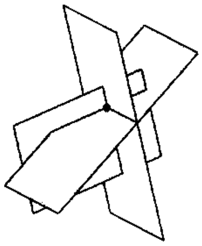


Fig 4

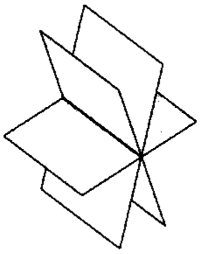


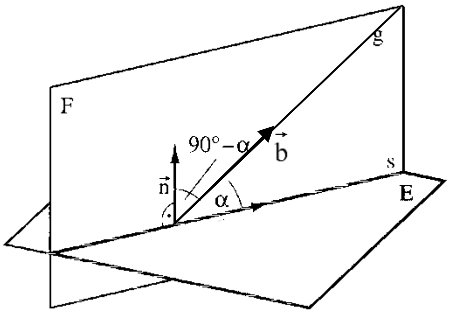
Fig 5

Exercises on planes Nos. 16 - 17

7.3.8. Cutting angle between plane and line

Cutting angle between plane and line:

The cutting angle between a line g with **direction vector** $\vec{b}$ and a plane E with **normal vector** $\vec{n}$ ist he **minimal** angle between g and any line s in E. The wanted line s is the intersecting line of the **cutting plane** F in the direction of $\vec{b}$ and $\vec{n}$ with E and is called the **trace** or **projection** (or **shadow**) of g on E. The cutting angle α can be obtained via the scalar product: $\cos(90° - \alpha) = \sin(\alpha) = \frac{\vec{n} * \vec{b}}{|\vec{n}| \cdot |\vec{b}|}$



Exercises on planes Nos. 18 - 19

7.3.9. Cutting angle between two planes

Cutting angle between two planes:

The cutting angle between the planes E_1 and E_2 is again the minimal angle between two vectors in E_1 and E_2. It is the angle between the **traces** s_1 and s_2 of the **cutting plane** F with directions $\vec{n}_1$ und $\vec{n}_2$. Since the **normal vectors** $\vec{n}_1$ und $\vec{n}_2$ are **perpendicular** to s_1 and s_2 the wanted angle can be simply obtained by $\cos(\alpha) = \frac{\vec{n}_1 * \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$

Exercises on planes No 20 - 21

7.3.10. Distance between plane and point

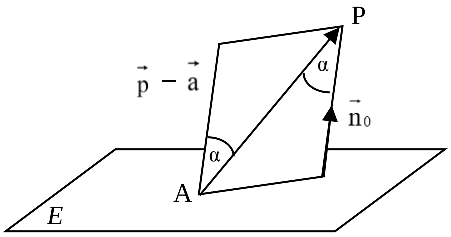
Distance between plane and point:

The distance d of a point P with position vector $\vec{p}$ from a plane

E: $(\vec{x} - \vec{a}) * \vec{n} = 0$ is

$d = \overline{AP} \cdot \cos(\alpha) = |(\vec{p} - \vec{a}) * \overrightarrow{n_0}|$

with the **normal unit vector** $\vec{n}_0 = \frac{\vec{n}}{|\vec{n}|}$.



Example:

Find the distance of the point P(1|3|−2) from the plane E: $x_1 - x_2 + 3x_3 = 2$.

Solution:

1. The **normal vector** $\vec{n} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ with length $|\vec{n}| = \sqrt{11}$ is shortened to length 1: $\vec{n_0} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{11}}\begin{pmatrix}1\\-1\\3\end{pmatrix}$.

2. Any solution of the plane equation can be chosen as **support vector**, e.g. $\vec{a} = \begin{pmatrix}2\\0\\0\end{pmatrix}$.

3. The distance is $d = |(\vec{p} - \vec{a}) * \vec{n_0}| = \left|\left(\begin{pmatrix}1\\3\\-2\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix}\right) * \frac{1}{\sqrt{11}}\begin{pmatrix}1\\-1\\3\end{pmatrix}\right| = \frac{1}{\sqrt{11}}\left|\begin{pmatrix}-1\\3\\-2\end{pmatrix} * \begin{pmatrix}1\\-1\\3\end{pmatrix}\right| = \frac{10}{\sqrt{11}}$.

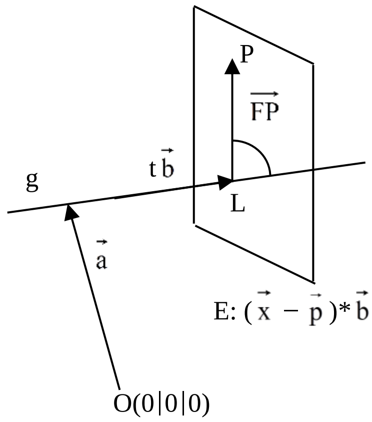
Exercises on planes No 22 - 26

7.3.11. Distance between a point and a line

Distance between a point and a line

The distance of a point P with position vector $\vec{p}$ form the line g: $\vec{x} = \vec{a} + t\vec{b}$ is $d = |\overrightarrow{LP}|$.

The perpendicular L of P on g is the intersection of the **auxiliary plane** E: $(\vec{x} - \vec{p}) * \vec{b} = 0$ **perpendicular to g through P**.



Example:

Find the distance of P(3|−1|2) from g: $\vec{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\0\\-2\end{pmatrix}$

Solutions:

1. The **auxiliary plane** is E: $(\vec{x} - \vec{p}) * \vec{b} = 0 \Leftrightarrow$ E: $\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix}\right) * \begin{pmatrix}1\\0\\-2\end{pmatrix} = 0 \Leftrightarrow$ E: $x_1 - 2x_3 = -1$.

2. The **perpendicular** $L = g \cap E$ can be obtained by substituting the line equation g: $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\0\\-2\end{pmatrix}$ in the plane equation E: $x_1 - 2x_3 = -1 \Rightarrow L = g \cap E$: $(1 + t) - 2(3 - 2t) = -1 \Leftrightarrow t = -\frac{3}{2}$. This yields $\overrightarrow{OL} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}0,5\\2\\6\end{pmatrix}$.

3. The distance is $d = |\overrightarrow{LP}| = |\overrightarrow{OP} - \overrightarrow{OL}| = \left|\begin{pmatrix}3\\-1\\2\end{pmatrix} - \begin{pmatrix}0,5\\2\\6\end{pmatrix}\right| = \frac{5}{4}\sqrt{5}$.

Exercises on planes No. 27

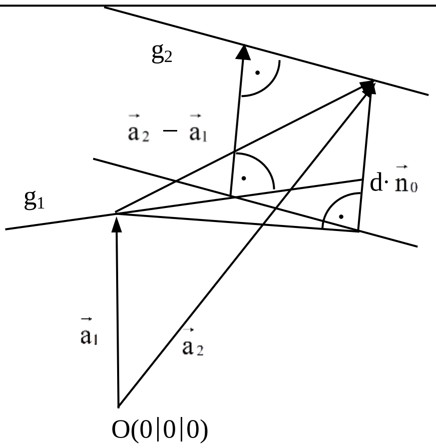
7.3.12. Distance between two skew lines

Distance between two skew lines:

The distance between the two lines g_1: $\vec{x} = \vec{a}_1 + r\vec{b}_1$ and g_2: $\vec{x} = \vec{a}_2 + s\vec{b}_2$ ist $d = |(\vec{a}_2 - \vec{a}_1) * \overrightarrow{n_0}|$

with the **common normal unit vector** $\overrightarrow{n_0} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|}$

which is perpendicular to g_1 as well as to g_2.



Example:

Find the distance between the lines g: $\vec{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\0\\-2\end{pmatrix}$ und h: $\vec{x} = \begin{pmatrix}3\\-1\\2\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}$.

Lösung:

The common normal unit vector is $\overrightarrow{n_0} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|} = \frac{1}{\sqrt{30}}\begin{pmatrix}2\\-5\\1\end{pmatrix}$.

Hence the distance is $d = |(\vec{a}_2 - \vec{a}_1) * \overrightarrow{n_0}| = \left|\left(\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix}\right) * \frac{1}{\sqrt{30}}\begin{pmatrix}2\\-5\\1\end{pmatrix}\right| = \frac{1}{\sqrt{30}}\left|\begin{pmatrix}-2\\3\\1\end{pmatrix} * \begin{pmatrix}2\\-5\\1\end{pmatrix}\right| = \frac{20}{\sqrt{30}}$

Exercises on planes Nos. 28 - 30